

COMS W3261: Computer Science Theory  
Pumping Lemma Solutions  
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Show that each of the following languages is not regular: (Hint: You may be able to avoid using the pumping lemma directly in a case or two.)

- a.  $\{0^l 1^m 2^{l+m} \mid l, m \geq 0\}$
- b.  $\{0^l 1^m 2^n \mid l, m, n \geq 0 \text{ such that } l + m \text{ is not equal to } n\}$
- c.  $\{0^{100+m} 1^n 0^n \mid m, n \geq 0\}$
- d.  $\{0^{n^3} \mid n \text{ is a non-negative number}\}$

This is the pumping lemma:

If  $L$  is a regular language, then there is a pumping number  $p$  such that, for all strings  $s$  in  $L$  of length at least  $p$ ,  $s$  can be split into three pieces,  $s = uvw$ , with  $v$  of length greater than zero,  $uv$  of length at most  $p$ , and for each  $n \geq 0$ ,  $uv^n w$  is in  $L$ .

Here is the contrapositive of the pumping lemma:

Suppose for all pumping numbers  $p$ , there exists a string  $s$  in  $L$  of length at least  $p$  such that, for all combinations of  $u, v, w$  such that  $s = uvw$ ,  $v$  of length greater than zero, and  $uv$  of length at most  $p$ , there exists an integer  $n$  such that  $uv^n w$  is not in  $L$ . Then  $L$  is not regular.

So, to prove that a language  $L$  is not regular imagine a game between you and the devil. You are trying to show that  $L$  isn't regular, but the devil is trying to undermine your efforts.

- (1) the devil chooses a number  $p$  and says: "I just picked a pumping number for  $L$ . Ha, ha!"
- (2) you choose a string  $s$  in  $L$  of length at least  $p$  and say: "I will show you that  $s$  isn't pumpable and the forces of evil shall be vanquished."
- (3) the devil splits  $s$  into  $uvw$ ,  $v$  of length greater than zero,  $uv$  of length at most  $p$  and says: "Show me that  $v$  can't be pumped, fool!!!"
- (4) you find  $n$  such that  $uv^n w$  is not in  $L$  and say "Your choices of  $p$  and  $v$  were fruitless. Good has prevailed over evil. Go back to hell!!!"

The point here is that your argument for defeating the devil has to be completely general and work against all possible choices of pumping numbers  $p$  and pumped substrings  $v$  (within the confines that  $v$  is non-empty and occurs in the first  $p$  letters of  $s$ ).

- a.  $\{0^l 1^m 2^{l+m} \mid l, m \geq 0\}$

Suppose this language is regular. Then there exists a pumping number  $p$ .

Consider the string  $s = 0^p 1^m 2^{p+m}$  ( $m$  can be any nonnegative integer, so you can choose  $m = p$  or  $m = 0$  if you like). Since  $uv$  must be at the beginning of  $s$  and cannot be longer than  $p$ , it must be within the  $0^p$ . So  $s$  breaks up as follows:

- $u = 0^i$ , where  $i \geq 0$
- $v = 0^j$ , where  $j > 0$
- $w = 0^k 1^m 2^{l+m}$ , where  $i + j + k = p$

Now consider the string  $uv^nw$ . This string has  $i + nj + k$  zeros,  $m$  ones, and  $p + m$  twos. The language requires  $i + nj + k + m = p + m$ . Plug in  $p = i + j + k$  and simplify the expression. You will get  $(n - 1)j = 0$ . Since  $j > 0$ , you can divide it out and you get  $n - 1 = 0$ . This condition is violated for all  $n$  not equal to 1. Thus there exists an  $n$  such that  $uv^nw$  is not in  $L$ , so  $L$  is not regular.

b.  $\{0^l1^m2^n \mid l, m, n \geq 0 \text{ such that } l + m \text{ is not equal to } n\}$

Suppose this language  $L$  is regular. Then its complement  $L^C$  is regular. Intersect  $L^C$  with the regular language  $0^*1^*2^*$ . The resulting language should be regular, since regular languages are closed under intersection. But the resulting language is the language from part a, which we have shown to be non-regular. This is a contradiction, therefore  $L$  must not have been regular. [This problem can also be done using the pumping lemma, if you consider the string  $0^p1^m2^{p+m+k}$ , where  $k = p! = p(p - 1)(p - 2)\dots(3)(2)(1)$  ]

c.  $\{0^{100+m}1^n0^n \mid m, n \geq 0\}$

To prove that this language  $L$  is not regular, we can show that its reverse  $L^R$  is not regular. The reverse of  $L$  is  $\{0^n1^n0^{100+m} \mid m, n \geq 0\}$ . Suppose  $L^R$  is regular. Then there exists a pumping number  $p$ . Consider the string  $s = 0^p1^p0^{100}$ . Since  $uv$  must be at the beginning of  $s$  and cannot be longer than  $p$ , it must be within the  $0^p$ . So  $s$  breaks up as follows:

- $u = 0^i$ , where  $i \geq 0$
- $v = 0^j$ , where  $j > 0$
- $w = 0^k1^p0^{100}$ , where  $i + j + k = p$

Now consider the string  $uv^nw$ . This string begins with  $i + nj + k$  zeros, followed by  $p$  ones. The language requires  $i + nj + k = p$ . Plug in  $p = i + j + k$  and simplify the expression. You will get  $(n - 1)j = 0$ . Since  $j > 0$ , you can divide it out and you get  $n - 1 = 0$ . This condition is violated for all  $n$  not equal to 1. Thus there exists an  $n$  such that  $uv^nw$  is not in  $L^R$ , so  $L^R$  is not regular. Thus  $L$  is not regular.

d.  $\{0^{n^3} \mid n \text{ is a non-negative number}\}$

Suppose this language is regular. Then there exists a pumping number  $p$ . Consider the string  $s = 0^{p^3}$ . It gets broken up as follows:

- $u = 0^i$ , where  $i \geq 0$
- $v = 0^j$ , where  $j > 0$
- $w = 0^k$ , where  $i + j + k = p^3$

Consider  $xy^n z$  and  $xy^{n+1} z$ . The lengths of these two strings differ by  $j$ . To be pumpable,  $xy^n z$  and  $xy^{n+1} z$  must both have perfect cube length. Suppose  $xy^n z$  has length  $a^3 = m$ . Then  $xy^{n+1} z$  has length  $m + j$ . The next perfect cube is  $(a + 1)^3 = a^3 + 3a^2 + 3a + 1$ , which is  $3a^2 + 3a + 1$  larger than  $a^3$ . So the next perfect cube is  $3m^{2/3} + 3m^{1/3} + 1$  larger than  $m$ . We can choose  $n$  to be any non-negative number, so choose  $n$  large enough so that  $j < 3m^{2/3} + 3m^{1/3} + 1$ . Then  $m + j$  cannot be a perfect cube because it is smaller than the next perfect cube. Therefore  $xy^{n+1} z$  is not in the language. Thus the language is not pumpable, and therefore not regular.