Consider the language of bitstrings:
\[ L = \{(0 \cup 1)^m0(0 \cup 1)^m0(0 \cup 1)^n0(0 \cup 1)^n | m, n \geq 0\} \]

Consider the grammar \( G \):
\[
S \rightarrow ZZ \\
Z \rightarrow 0 | AZA \\
A \rightarrow 0 | 1
\]

Prove that \( G \) generates the language \( L \):

To prove that \( G \) generates \( L \), i.e. \( L(G) = L \), you have to show two things:

1. Prove \( L(G) \subset L \), i.e. all strings derivable from \( G \) are in \( L \)
2. Prove \( L \subset L(G) \), i.e. every string in \( L \) has a derivation in \( G \)

Notice that the first production of \( G \) generates a concatenation of two \( Z \)'s, each of which will generate a string. Also notice that strings in \( L \) are the concatenation of two strings in the language \( L' = \{(0|1)^i0(0|1)^i\} \), i.e. \( L = L'L' \). Thus it is enough to show that \( L(Z) = L' \) where \( L(Z) \) denotes the set of terminal strings generated from the variable \( Z \).

**Lemma 0.1.** \( L(Z) \subset L' \)

**Proof.** For the base case, there is one string with derivation of length 1, that is the string 0. It is generated by the derivation \( Z \Rightarrow 0 \). The string 0 is in \( L' \) because it is of the form \( (0|1)^k0(0|1)^k \) with \( k = 0 \).

Now make the inductive hypothesis: If \( Z \Rightarrow^* w \) in less than \( n \) steps, then \( w \in L' \).

Let \( w \) be a string such that \( Z \Rightarrow^* w \) in \( n \) steps. Since the derivation has length greater than 1, it must use the production \( Z \Rightarrow AZA \). The inner \( Z \) generates a string \( w' \) in less than \( n \) steps. By the inductive hypothesis, \( w' \in L' \), i.e. \( w' = (0|1)^k0(0|1)^k \) for some \( k \). Since each \( A \) produces a 0 or 1, the result is that \( Z \Rightarrow AZA \Rightarrow (0|1)w'(0|1) = (0|1)^k+10(0|1)^k+1 \). Therefore \( w \in L' \). \( \square \)

**Lemma 0.2.** \( L' \subset L(Z) \)

**Proof.** For the base case, the shortest string in \( L' \) is the string 0. This is generated by \( Z \Rightarrow 0 \).

Now make the inductive hypothesis: If \( |w| \leq n \) and \( w \in L' \), then \( Z \Rightarrow w \).

Let \( w \) be any string of the form \( (0|1)^{n/2}0(0|1)^{n/2} \) (here \( n/2 \) denotes integer division). This string can be broken up into \((0|1)(0|1)^{n/2-1}0(0|1)^{n/2-1}(0|1)\). The inner string \( w' \) of the form \((0|1)^{n/2-1}0(0|1)^{n/2-1}\) has length at most \( n - 1 \) and is
of the form $(0|1)^k 0(0|1)^k$, i.e. $w' \in L'$. Thus we may apply the induction hypothesis and assume that $w'$ is generated by $Z$, i.e. $Z \Rightarrow^* w'$. So $Z \Rightarrow AZA \Rightarrow^* (0|1)w'(0|1) = w$. Thus, we have extended the inductive hypothesis to the next higher length and proved the theorem. \qed