There are various symbols

Symbols may be identical, even when drawn slightly differently. Other symbols are distinct. Symbols arranged in a horizontal sequence are "words," "strings," or "expressions." Symbols may represent values, operations, or relationships. Some symbols are treated as variables that represent other symbols. The meaning of an expression with variables depends on the variables' values.
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The meaning of an expression with variables depends on the variables’ values
Judgment

A judgment is an assertion about one or more things, typically membership in a set.

0 ∈ ℤ
0 is a member of the set of natural numbers

n nat
n is a member of the set of natural numbers

1 + 2 expr
1 + 2 is in the set of expressions

τ type
τ is in the set of types

e : τ
Expression e has type τ

sum(n₁, n₂, n₃)
Adding n₁ and n₂ gives n₃

n₁ + n₂ = n₃
Adding n₁ and n₂ gives n₃

Prefix; infix; and suffix syntax
Inference Rule

Premises: Judgments $\rightarrow J_1 J_2 \ldots J_k$
Conclusion: A Judgment $\rightarrow \overline{J}$

“If all the premises hold, the conclusion follows”

The Natural Numbers Defined Inductively by Two Inference Rules (Peano)

Axiom $\rightarrow 0 \in \mathbb{N}$ zero

Judgments: $a \in \mathbb{N}$

Variables: $a \leftarrow$ Sequences of symbols

Symbols: $0 \quad \text{succ}(\quad)$

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>succ(0)</th>
<th>succ(succ(0))</th>
<th>succ(succ(succ(0)))</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Inference Rule

Premises: Judgments → \[ J_1 \quad J_2 \quad \cdots \quad J_k \]
Conclusion: A Judgment → \[ J \]

“If all the premises hold, the conclusion follows”

The Natural Numbers Defined Inductively by Two Inference Rules (Peano)

\[
\begin{align*}
0 & \in \mathbb{N} \quad \text{zero} \\
\text{succ}(a) & \in \mathbb{N} \quad \text{successor}
\end{align*}
\]

Judgments: \( a \in \mathbb{N} \)
Variables: \( a \leftarrow \text{Sequences of symbols} \)
Symbols: \( 0 \quad \text{succ}() \)

\[
\begin{align*}
0 & \quad 0 \\
\text{succ}(0) & \quad 1 \\
\text{succ}(\text{succ}(0)) & \quad 2 \\
\text{succ}(\text{succ}(\text{succ}(0))) & \quad 3 \\
\text{succ}(\text{succ}(\text{succ}(\text{succ}(0)))) & \quad 4
\end{align*}
\]
Inference Rule

Premises: Judgments → $J_1 \ J_2 \ \cdots \ J_k$  
Conclusion: A Judgment → $J$  

“If all the premises hold, the conclusion follows”

The Natural Numbers Defined Inductively by Two Inference Rules (Peano)

- $\theta \in \mathbb{N}$ zero
- $a \in \mathbb{N}$ successor

Technically a scheme

Scheme: pattern with variables: replacing $a$ consistently gives a rule

- $\theta \in \mathbb{N}$
- $\text{succ}(\theta) \in \mathbb{N}$
- $\text{true} \in \mathbb{N}$
- $\text{succ}(\text{true}) \in \mathbb{N}$
- $\text{succ}(\text{succ}(\theta)) \in \mathbb{N}$

Which are variables? Values constrained? Variable scope: a single rule

Consistent replacement only:

- $\text{foo} \in \mathbb{N}$
- $\text{succ}(\text{bar}) \in \mathbb{N}$

is not a rule
Inference Rule

Premises: Judgments → $\frac{J_1 \ J_2 \ \ldots \ \ J_k}{J}$ Rule-Name

Conclusion: A Judgment → “If all the premises hold, the conclusion follows”

The Natural Numbers Defined Inductively by Two Inference Rules (Peano)

\[
\begin{align*}
\text{zero} & \quad 0 \in \mathbb{N} \\
\text{successor} & \quad a \in \mathbb{N} \\
\text{succ(succ(succ(0)))} & \quad \text{successor}
\end{align*}
\]

Is succ(succ(succ(0)))) a.k.a. 3 a natural number? A forward derivation

\[
\begin{align*}
\text{zero} & \quad 0 \in \mathbb{N}
\end{align*}
\]
Inference Rule

Premises: Judgments → $J_1 \ J_2 \ ... \ J_k$  
Conclusion: A Judgment → $\frac{J}{J}$  
Rule-Name

“If all the premises hold, the conclusion follows”

The Natural Numbers Defined Inductively by Two Inference Rules (Peano)

\[
\begin{align*}
\text{zero:} & \quad 0 \in \mathbb{N} \\
\text{successor:} & \quad a \in \mathbb{N} \implies \text{succ}(a) \in \mathbb{N}
\end{align*}
\]

Is succ(succ(succ(0))) a.k.a. 3 a natural number? A forward derivation

\[
\begin{align*}
\text{zero:} & \quad 0 \in \mathbb{N} \\
\text{successor:} & \quad \text{succ}(0) \\
\text{choose} & \quad a = 0
\end{align*}
\]
Inference Rule

Premises: Judgments → $J_1 J_2 \cdots J_k$ → Conclusion: A Judgment → $J$

“If all the premises hold, the conclusion follows”

The Natural Numbers Defined Inductively by Two Inference Rules (Peano)

$$
\begin{align*}
\text{zero} & \equiv 0 \\
\text{successor} & \equiv \text{succ}(a) \\
\end{align*}
$$

Is $\text{succ} (\text{succ} (\text{succ} (\text{zero})))$ a.k.a. 3 a natural number? A forward derivation

$$
\begin{align*}
\text{zero} & \equiv 0 \\
\text{successor} & \equiv \text{succ}(a) \\
\text{succ}(\text{zero}) & \equiv \text{succ}(0) \\
\text{succ}(\text{succ}(\text{zero})) & \equiv \text{succ}(\text{succ}(0)) \\
\end{align*}
$$

← choose $a = \text{succ}(0)$
Inference Rule

Premises: Judgments → $J_1, J_2, \ldots, J_k$ Rule-Name
Conclusion: A Judgment → $J$

“If all the premises hold, the conclusion follows”

The Natural Numbers Defined Inductively by Two Inference Rules (Peano)

\[
\begin{align*}
\text{zero} & : 0 \in \mathbb{N} \\
\text{successor} & : a \in \mathbb{N} \Rightarrow \text{succ}(a) \in \mathbb{N}
\end{align*}
\]

Is $\text{succ}((\text{succ}(\text{succ}(0))))$ a.k.a. 3 a natural number? A forward derivation

\[
\begin{align*}
\text{zero} & : 0 \in \mathbb{N} \\
\text{successor} & : \text{succ}(0) \\
\text{successor} & : \text{succ}(\text{succ}(0)) \\
\text{successor} & : \text{succ}(\text{succ}(\text{succ}(0)))
\end{align*}
\]
The Natural Numbers

\[
\begin{align*}
\text{zero} & \quad & 0 \in \mathbb{N} \\
\text{successor} & \quad & a \in \mathbb{N} \quad \text{succ}(a) \in \mathbb{N}
\end{align*}
\]

\[
\text{zeroIsN} :: \text{String} \rightarrow \text{Bool}
\]

\[
\text{successorIsN} :: \text{String} \rightarrow \text{Bool}
\]

String is inefficient, but let’s focus on correctness first
The Natural Numbers

\[
\begin{array}{c}
\text{zero} \\
\emptyset \in \mathbb{N} \\
\text{successor} \\
\text{succ}(a) \in \mathbb{N}
\end{array}
\]

zeroIsN :: String -> Bool
zeroIsN "0" = True
zeroIsN _ = False  -- Default case

successorIsN :: String -> Bool
The Natural Numbers

\[
\begin{align*}
0 & \in \mathbb{N} \\
\text{successor}(a) & \in \mathbb{N}
\end{align*}
\]

import Data.List (stripPrefix) -- stripPrefix :: String -> String -> Maybe String

zeroIsN :: String -> Bool
zeroIsN "0" = True
zeroIsN _ = False -- Default case

successorIsN :: String -> Bool -- Construct a Reverse Derivation
successorIsN s = case stripPrefix "succ(\(s\) of
  Just aa@(\_:_\) -> last aa == ')' &&
  let a = init aa in
    zeroIsN a || successorIsN a -- Try both
  _ -> False

-- Of the form succ(...)?
-- Prohibit the empty string
-- Get all but last character
The Natural Numbers

\[
\begin{align*}
\text{zero} & \equiv 0 \\
\text{successor} & \equiv \text{succ}(a) \\
0 & \in \mathbb{N} \\
\text{succ}(a) & \in \mathbb{N}
\end{align*}
\]

import Data.List (stripPrefix)

zeroIsN :: String -> Bool
zeroIsN "0" = True
zeroIsN _   = False

successorIsN :: String -> Bool
successorIsN s = case match "succ(" ")" s of
    Just a  -> zeroIsN a || successorIsN a
    _      -> False

match :: String -> String -> String -> Maybe String -- Helper function
match pre suff s = do a' <- stripPrefix pre s -- Stops at Nothing
    reverse <$> stripPrefix (reverse suff) (reverse a')
The Natural Numbers

\[
\begin{align*}
\text{zero} & \quad 0 \in \mathbb{N} \\
\text{successor} & \quad a \in \mathbb{N} \Rightarrow \text{succ}(a) \in \mathbb{N}
\end{align*}
\]

import Data.List (stripPrefix)

isNat :: String -> Bool

-- Merge the two rules
isNat "0" = True

-- zero rule
isNat s = case match "succ(" ")" s of
  Just a -> isNat a
  Nothing -> False

-- successor rule

Just a -> isNat a

-- Only one thing to check

Nothing -> False

match :: String -> String -> String -> Maybe String

match pre suff s = do a' <- stripPrefix pre s
                     reverse <$> stripPrefix (reverse suff) (reverse a')
The Natural Numbers

\[ 0 \in \mathbb{N} \quad \text{successor} \quad \text{succ}(a) \in \mathbb{N} \]

\[
\begin{align*}
\text{data} & \quad \text{Nat} = \text{Zero} \mid \text{Succ Nat} \quad \text{-- Algebraic data type: either “Zero” or “Succ n”} \\
\text{zeroIsN} & \quad :: \text{Nat} \to \text{Bool} \\
\text{zeroIsN Zero} & = \text{True} \\
\text{zeroIsN _} & = \text{False} \\
\text{successorIsN} & \quad :: \text{Nat} \to \text{Bool} \\
\text{successorIsN (Succ a)} & = \text{zeroIsN a} \mid\mid \text{successorIsN a} \quad \text{-- Try both} \\
\text{successorIsN _} & = \text{False}
\end{align*}
\]
The Natural Numbers

\[ \begin{align*}
\text{zero} & : 0 \in \mathbb{N} \\
\text{successor} & : a \in \mathbb{N} \quad \Rightarrow \quad \text{succ}(a) \in \mathbb{N}
\end{align*} \]

data Nat = Zero | Succ Nat

isNat :: Nat -> \text{Bool}
isNat Zero = \text{True} \quad -- \text{zero rule}
isNat (Succ a) = isNat a \quad -- \text{successor rule}

isNat is trivial; Haskell’s type system enforces it for us
Equality of Natural Numbers as an Inductive Definition

\[
\begin{align*}
\text{equalzero} & \quad \text{eq}(\emptyset, 0) \\
\text{eq(a, b)} & \quad \text{eq}(\text{succ}(a), \text{succ}(b))
\end{align*}
\]

Judgements: \( \text{eq}(n_1, n_2) \) “\( n_1 \) and \( n_2 \) are equal” ← a relation/a set of pairs
Variables: \( a \quad b \)
Symbols: \( \emptyset \quad \text{succ}(\quad) \)

Is 3 = 3? A reverse derivation

\[
\text{eq}(\text{succ}(\text{succ}(\text{succ}(\emptyset))), \text{succ}(\text{succ}(\text{succ}(\emptyset))))
\]
Equality of Natural Numbers as an Inductive Definition

\[
\begin{align*}
\text{equalzero} & \quad \text{eq}(0, 0) \\
\text{equal} & \quad \frac{\text{eq}(a, b)}{\text{eq}(\text{succ}(a), \text{succ}(b))}
\end{align*}
\]

Judgements: \( \text{eq}(n_1, n_2) \) "\( n_1 \) and \( n_2 \) are equal"

Variables: \( a \quad b \)

Symbols: \( \text{0} \quad \text{succ}() \)

Is 3 = 3? A reverse derivation

\[
\begin{align*}
\text{eq}(\text{succ}(\text{succ}(\text{succ}(0))),\text{succ}(\text{succ}(\text{succ}(0)))) & \quad \text{equal} \\
\text{eq}(\text{succ}(\text{succ}(\text{succ}(0))),\text{succ}(\text{succ}(\text{succ}(0)))) & \quad \text{equal} \\
\text{eq}(\text{succ}(\text{succ}(0)),\text{succ}(\text{succ}(0))) & \quad \text{equal} \\
\text{eq}(\text{0}, \text{succ}(0)) & \quad \text{equal} \\
\text{eq}(\text{0}, \text{0}) & \quad \text{equalzero}
\end{align*}
\]
Equality of Natural Numbers as an Inductive Definition

\[
\begin{align*}
\text{eq}(0,0) & \quad \text{equalzero} \\
\text{eq}(\text{succ}(a),\text{succ}(b)) & \quad \text{equal}
\end{align*}
\]

Judgements: \( \text{eq}(n_1, n_2) \) “\( n_1 \) and \( n_2 \) are equal”

Variables: \( a \quad b \)

Symbols: \( 0 \quad \text{succ}(\quad) \)

Is \( 3 = 3? \) A reverse derivation

\[
\begin{align*}
\text{eq}(\text{succ}(0), \text{succ}(0)) & \quad \text{equal} \\
\text{eq}(\text{succ}(\text{succ}(0)), \text{succ}(\text{succ}(0))) & \quad \text{equal} \\
\text{eq}(\text{succ}(\text{succ}(\text{succ}(0))), \text{succ}(\text{succ}(\text{succ}(0)))) & \quad \text{equal}
\end{align*}
\]
Equality of Natural Numbers as an Inductive Definition

<table>
<thead>
<tr>
<th>eq(0, 0)</th>
<th>eq(a, b)</th>
</tr>
</thead>
<tbody>
<tr>
<td>equalzero</td>
<td>equalsucc</td>
</tr>
</tbody>
</table>

Judgements: \( \text{eq}(n_1, n_2) \) “\( n_1 \) and \( n_2 \) are equal”

Variables: \( a \quad b \)

Symbols: \( 0 \quad \text{succ}(\ ) \)

Is 3 = 3? A reverse derivation

\[
\begin{align*}
\text{eq}(0, 0) & \quad \text{equalzero} \\
\text{eq}(\text{succ}(0), \text{succ}(0)) & \quad \text{equalsucc} \\
\text{eq}(\text{succ}(\text{succ}(0)), \text{succ}(\text{succ}(0))) & \quad \text{equalsucc} \\
\text{eq}(\text{succ}(\text{succ}(\text{succ}(0)))) & \quad \text{equalsucc} \\
\end{align*}
\]
Equality of Natural Numbers as an Inductive Definition

\[
\begin{align*}
\text{equalzero:} & \quad \text{eq}(\emptyset, \emptyset) \\
\text{eq}(a, b): & \quad \text{eq}(\text{succ}(a), \text{succ}(b))
\end{align*}
\]

Judgements: \( \text{eq}(n_1, n_2) \) “\( n_1 \) and \( n_2 \) are equal”

Variables: \( a, b \)

Symbols: \( \emptyset, \text{succ}(\quad) \)

Is 3 = 3? A reverse derivation

\[
\begin{align*}
\text{eq}(\emptyset, \emptyset) & \quad \text{equalzero} \\
\text{eq}(\text{succ}(\emptyset), \text{succ}(\emptyset)) & \quad \text{equal} \\
\text{eq}(\text{succ}(\text{succ}(\emptyset)), \text{succ}(\text{succ}(\emptyset))) & \quad \text{equal} \\
\text{eq}(\text{succ}(\text{succ}(\text{succ}(\emptyset))), \text{succ}(\text{succ}(\text{succ}(\emptyset)))) & \quad \text{equal}
\end{align*}
\]

We are stuck: neither rule applies, so 1 \( \neq \) 2
Equality of Natural Numbers as an Inductive Definition

\[
\begin{align*}
\text{equalzero} & : \quad \text{eq}(0, 0) \\
\text{equal} & : \quad \text{eq}(a, b) \rightarrow \text{eq}(\text{succ}(a), \text{succ}(b))
\end{align*}
\]

Judgements:  \( \text{eq}(n_1, n_2) \)  “\( n_1 \) and \( n_2 \) are equal”

Variables:  \( a \quad b \)

Symbols:  \( 0 \quad \text{succ}(\quad) \)

Is \( 3 = 3? \) A reverse derivation

\[
\begin{align*}
\text{eq}(\quad,\quad) & \quad \text{equalzero} \\
\text{eq}(\quad,\quad) & \quad \text{equal} \\
\text{eq}(\quad,\quad) & \quad \text{equal} \\
\text{eq}(\quad,\quad) & \quad \text{equal} \\
\text{eq}(\quad,\quad) & \quad \text{equal}
\end{align*}
\]

Is \( 1 = 2? \)

\[
\begin{align*}
\text{eq}(\quad,\quad) & \quad \text{equal}
\end{align*}
\]
Equality of Natural Numbers as an Inductive Definition

\[
\begin{align*}
\text{eq}(\theta, \theta) & \quad \text{equalzero} \\
\text{eq}(\text{succ}(a), \text{succ}(b)) & \quad \text{equal}
\end{align*}
\]

Judgements: \( \text{eq}(n_1, n_2) \) "\( n_1 \) and \( n_2 \) are equal"

Variables: \( a \quad b \)

Symbols: \( \theta \quad \text{succ}(\quad) \)

Is 3 = 3? A reverse derivation

\[
\begin{align*}
\text{eq}(\theta, \theta) & \quad \text{equalzero} \\
\text{eq}(\text{succ}(\theta), \text{succ}(\theta)) & \quad \text{equal} \\
\text{eq}(\text{succ}(\text{succ}(\theta)), \text{succ}(\text{succ}(\theta))) & \quad \text{equal} \\
\text{eq}(\text{succ}(\text{succ}(\text{succ}(\theta))), \text{succ}(\text{succ}(\text{succ}(\theta)))) & \quad \text{equal}
\end{align*}
\]

Is 1 = 2?

\[
\begin{align*}
\text{eq}(\theta, \text{succ}(\theta)) & \quad \text{equal} \\
\text{eq}(\text{succ}(\theta), \text{succ}(\text{succ}(\theta))) & \quad \text{equal}
\end{align*}
\]
### Equality of Natural Numbers as an Inductive Definition

\[
\begin{align*}
\text{eq}(0, 0) & \quad \text{equalzero} \\
\text{eq}(\text{succ}(a), \text{succ}(b)) & \quad \text{equal}
\end{align*}
\]

Judgements: \( \text{eq}(n_1, n_2) \)  “\( n_1 \) and \( n_2 \) are equal”

Variables:  \( a \quad b \)

Symbols:  \( 0 \quad \text{succ}(\phantom{0}) \)

---

**Is 3 = 3? A reverse derivation**

\[
\begin{align*}
\text{eq}(0, 0) & \quad \text{equalzero} \\
\text{eq}(\text{succ}(0), \text{succ}(0)) & \quad \text{equal} \\
\text{eq}(\text{succ}(\text{succ}(0)), \text{succ}(\text{succ}(0))) & \quad \text{equal} \\
\text{eq}(\text{succ}(\text{succ}(\text{succ}(0))), \text{succ}(\text{succ}(\text{succ}(0)))) & \quad \text{equal}
\end{align*}
\]

---

**Is 1 = 2?**

\[
\begin{align*}
\text{eq}(0, \text{succ}(0)) & \quad \text{equal} \\
\text{eq}(\text{succ}(0), \text{succ}(\text{succ}(0))) & \quad \text{equal}
\end{align*}
\]

We are stuck: neither rule applies, so \( 1 \neq 2 \)
Equality of Natural Numbers as an Inductive Definition

\[
\begin{align*}
\text{equalzero} & : 0 = 0 \\
\text{equal} & : a = b \implies \text{succ}(a) = \text{succ}(b)
\end{align*}
\]

Judgements: \( n_1 = n_2 \) \( \text{“} n_1 \text{ and } n_2 \text{ are equal”} \) ← a relation/a set of pairs

Variables: \( a \quad b \)

Symbols: \( 0 \quad \text{succ}(\quad ) \)

Is 3 = 3?

\[
\begin{align*}
\text{equalzero} & : 0 = 0 \\
\text{equal} & : \text{succ}(0) = \text{succ}(0) \\
\text{equal} & : \text{succ}(\text{succ}(0)) = \text{succ}(\text{succ}(0)) \\
\text{equal} & : \text{succ}(\text{succ}(\text{succ}(0))) = \text{succ}(\text{succ}(\text{succ}(0)))
\end{align*}
\]

Is 1 = 2?

\[
\begin{align*}
0 & = \text{succ}(0) \\
\text{equal} & : \text{succ}(0) = \text{succ}(\text{succ}(0))
\end{align*}
\]

We are stuck: neither rule applies, so 1 \( \neq \) 2
Equality of Natural Numbers as an Inductive Definition

\[
\begin{align*}
\theta = \theta & \quad \text{equalzero} \\
\text{succ}(a) = \text{succ}(b) & \quad \text{equal}
\end{align*}
\]

\[
\begin{align*}
\text{data Nat} & = \text{Zero} \mid \text{Succ Nat} \\
\text{natEqual} :: \text{Nat} \rightarrow \text{Nat} & \rightarrow \text{Bool} \\
\text{natEqual Zero Zero} & = \text{True} \quad \text{-- equalzero rule} \\
\text{natEqual (Succ a) (Succ b)} & = \text{natEqual a b} \quad \text{-- equal rule} \\
\text{natEqual _ _} & = \text{False}
\end{align*}
\]

Again: single function because only one rule may ever match
Equality of Natural Numbers as an Inductive Definition

\[
\begin{align*}
\text{equalzero} & : 0 = 0 \\
\text{equal} & : \text{succ}(a) = \text{succ}(b)
\end{align*}
\]

data Nat = Zero | Succ Nat

\[\text{deriving Eq}\]

This Haskell’s default implementation of == for algebraic data types
Addition as an Inductive Definition

<table>
<thead>
<tr>
<th>Rule</th>
<th>Premise</th>
<th>Conclusion</th>
</tr>
</thead>
<tbody>
<tr>
<td>addzero</td>
<td>$b \in \mathbb{N}$</td>
<td>$\text{sum}(0, b, b)$</td>
</tr>
<tr>
<td>add</td>
<td>$\text{sum}(a, b, c)$</td>
<td>$\text{sum}(\text{succ}(a), b, \text{succ}(c))$</td>
</tr>
</tbody>
</table>

Judgments: $n \in \mathbb{N}$ \hspace{1cm} $\text{sum}(n_1, n_2, n_3) \leftarrow \text{a relation/a set of triples}$

Variables: $a \hspace{0.5cm} b \hspace{0.5cm} c$

Symbols: $0 \hspace{0.5cm} \text{succ}( \hspace{0.5cm} )$
Addition as an Inductive Definition

\[
\frac{b \in \mathbb{N}}{\text{sum}(\emptyset, b, b)} \quad \frac{\text{sum}(a, b, c)}{\text{sum}(\text{succ}(a), b, \text{succ}(c))}
\]

Judgments: \( n \in \mathbb{N} \quad \text{sum}(n_1, n_2, n_3) \)

Variables: \( a \quad b \quad c \)

Symbols: \( 0 \quad \text{succ}( \quad ) \)

Is \( 2 + 1 = 3? \)

\[
\text{sum}(\text{succ}(\text{succ}(\emptyset)), \text{succ}(\emptyset), \text{succ}(\text{succ}(\text{succ}(\emptyset))))
\]
Addition as an Inductive Definition

\[
\begin{align*}
    b \in \mathbb{N} & \quad \frac{}{\text{sum}(0, b, b)} \quad \text{addzero} \\
    \text{sum}(a, b, c) & \quad \frac{}{\text{sum}((\text{succ}(a), b, \text{succ}(c)))} \quad \text{add}
\end{align*}
\]

Judgments: \( n \in \mathbb{N} \quad \text{sum}(n_1, n_2, n_3) \)

Variables: \( a \quad b \quad c \)

Symbols: \( 0 \quad \text{succ}(\quad) \)

Is \( 2 + 1 = 3? \)

\[
\begin{align*}
    \text{sum}( & \quad \text{succ}(\quad) \quad , \quad \text{succ}(\quad) \quad , \quad \text{succ}(\text{succ}(\quad))) \quad \text{add} \\
    \text{sum}( & \quad \text{succ}(\text{succ}(\quad)) \quad , \quad \text{succ}(\quad) \quad , \quad \text{succ}(\text{succ}(\text{succ}(\quad))))
\end{align*}
\]
Addition as an Inductive Definition

\[
\begin{align*}
\text{addzero} & \quad b \in \mathbb{N} \\
\text{sum}(\emptyset, b, b) & \quad \text{sum}(a, b, c) \\
\text{add} & \quad \text{sum}(\text{succ}(a), b, \text{succ}(c))
\end{align*}
\]

Judgments: \( n \in \mathbb{N} \quad \text{sum}(n_1, n_2, n_3) \)

Variables: \( a \quad b \quad c \)

Symbols: \( \emptyset \quad \text{succ}(\quad) \)

Is 2 + 1 = 3?

\[
\begin{align*}
\text{sum}(0, \text{succ}(0), \text{succ}(0)) & \quad \text{add} \\
\text{sum}(\text{succ}(0), \text{succ}(0), \text{succ}(\text{succ}(0))) & \quad \text{add} \\
\text{sum}(\text{succ}(\text{succ}(0)), \text{succ}(0), \text{succ}(\text{succ}(\text{succ}(0)))) & \quad \text{add}
\end{align*}
\]
Addition as an Inductive Definition

\[
\begin{align*}
\text{addzero} & : \quad b \in \mathbb{N} \\
\text{sum} & : \quad \text{addzero} \quad \frac{\text{sum}(0, b, b)}{	ext{sum}(\text{succ}(a), b, \text{succ}(c))} \quad \text{add}
\end{align*}
\]

Judgments: \( n \in \mathbb{N} \quad \text{sum}(n_1, n_2, n_3) \)

Variables: \( a \quad b \quad c \)

Symbols: \( 0 \quad \text{succ}() \)

Is 2 + 1 = 3?

\[
\begin{align*}
\text{addzero} & : \quad \text{succ}(\text{0}) \in \mathbb{N} \\
\text{add} & : \quad \frac{\text{sum}(0, \text{succ}(\text{0}), \text{succ}(\text{0}))}{\text{add}} \\
\text{add} & : \quad \frac{\text{sum}(\text{succ}(\text{0}), \text{succ}(\text{0}), \text{succ}(\text{succ}(\text{0})))}{\text{add}} \\
\text{add} & : \quad \frac{\text{sum}(\text{succ}(\text{succ}(\text{0})), \text{succ}(\text{0}), \text{succ}(\text{succ}(\text{succ}(\text{0}))))}{\text{add}}
\end{align*}
\]
Addition as an Inductive Definition

\[ b \in \mathbb{N} \]
\[ \text{sum}(\emptyset, b, b) \text{addzero} \]
\[ \text{sum}(a, b, c) \text{add} \]
\[ \text{sum}(\text{succ}(a), b, \text{succ}(c)) \]

Judgments: \( n \in \mathbb{N} \quad \text{sum}(n_1, n_2, n_3) \)

Variables: \( a \quad b \quad c \)

Symbols: \( 0 \quad \text{succ}(\quad) \)

Is \( 2 + 1 = 3? \)

\[ 0 \in \mathbb{N} \quad \text{successor} \]
\[ \text{succ}(0) \in \mathbb{N} \]
\[ \text{sum}(0, \text{succ}(0), \text{succ}(0)) \text{addzero} \]
\[ \text{sum}(\text{succ}(0), \text{succ}(0), \text{succ}(\text{succ}(0))) \text{add} \]
\[ \text{sum}(\text{succ}(\text{succ}(0)), \text{succ}(0), \text{succ}(\text{succ}(\text{succ}(0)))) \text{add} \]
Addition as an Inductive Definition

\[
\begin{align*}
  b \in \mathbb{N} & \quad \frac{}{\text{addzero} \quad \text{sum}(0, b, b)} \\
  \text{sum}(a, b, c) & \quad \frac{}{\text{add} \quad \text{sum}(\text{succ}(a), b, \text{succ}(c))}
\end{align*}
\]

Judgments: \( n \in \mathbb{N} \quad \text{sum}(n_1, n_2, n_3) \)

Variables: \( a \quad b \quad c \)

Symbols: \( 0 \quad \text{succ}() \)

Is \( 2 + 1 = 3? \)

\[
\begin{align*}
  \text{zero} & \quad \frac{}{0 \in \mathbb{N}} \\
  \text{successor} & \quad \frac{}{\text{succ}(0) \in \mathbb{N}} \\
  \text{addzero} & \quad \frac{}{\text{sum}(0, \text{succ}(0), \text{succ}(0))} \\
  \text{add} & \quad \frac{}{\text{sum}(\text{succ}(0), \text{succ}(0), \text{succ}(\text{succ}(0)))} \\
  \text{add} & \quad \frac{}{\text{sum}(	ext{succ}(\text{succ}(0)), \text{succ}(0), \text{succ}(\text{succ}(\text{succ}(0))))}
\end{align*}
\]
Addition as an Inductive Definition

\[
\begin{align*}
    b \in \mathbb{N} & \quad \text{addzero} \\
    \theta + b = b & \\
    a + b = c & \quad \text{add} \\
    \text{succ}(a) + b = \text{succ}(c)
\end{align*}
\]

Judgments: \( n \in \mathbb{N} \quad n_1 + n_2 = n_3 \quad \leftarrow \text{a relation/a set of triples} \)

Variables: \( a \quad b \quad c \)

Symbols: \( \theta \quad \text{succ}(\ ) \)

Is \( 2 + 1 = 3? \)

\[
\begin{align*}
    \theta \in \mathbb{N} & \quad \text{zero} \\
    \text{succ}(\theta) \in \mathbb{N} & \quad \text{successor} \\
    \theta + \text{succ}(\theta) = \text{succ}(\theta) & \quad \text{addzero} \\
    \text{succ}(\theta) + \text{succ}(\theta) = \text{succ}(\text{succ}(\theta)) & \quad \text{add} \\
    \text{succ}(\text{succ}(\theta)) + \text{succ}(\theta) = \text{succ}(\text{succ}(\text{succ}(\theta))) & \quad \text{add}
\end{align*}
\]
Addition as an Inductive Definition

\[
\begin{align*}
\frac{b \in \mathbb{N}}{\theta + b = b} & \quad \text{addzero} \\
\frac{a + b = c}{\text{succ}(a) + b = \text{succ}(c)} & \quad \text{add}
\end{align*}
\]

```haskell
data Nat = Zero | Succ Nat
  deriving Eq
```

```haskell
sumsTo :: Nat -> Nat -> Nat -> Bool -- Is \(a + b = c\)?
sumsTo Zero b b' | b == b' = True -- addzero rule
sumsTo (Succ a) b (Succ c) = sumsTo a b c -- add rule
sumsTo _ _ _ = False -- E.g., (Succ a) _ Zero
```

No need to check whether \(b \in \mathbb{N}\): the types enforce this
Haskell patterns can’t check for equality like \(\text{sumsTo} \ Zero \ b \ b\), so I added guard \(b \ == \ 'b\)
Rather awkward to ask “is this it?”
Addition as an Inductive Definition

\[
\begin{align*}
  b \in \mathbb{N} & \quad \text{addzero} \\
  \theta + b &= b & a + b &= c & \text{add} \\
  \text{succ}(a) + b &= \text{succ}(c)
\end{align*}
\]

data Nat = Zero | Succ Nat

deriving (Eq, Show)

addNat :: Nat -> Nat -> Nat

\[
\begin{align*}
   \text{addNat Zero } b &= b & \text{-- addzero rule} \\
   \text{addNat (Succ a) } b &= \text{Succ (addNat a b)} & \text{-- add rule}
\end{align*}
\]

The dataflow makes this easy and it's obviously a total function
Addition as an Inductive Definition

\[
\begin{align*}
\text{addzero} : & \quad b \in \mathbb{N} \\
\hline
\theta + b & = b
\end{align*}
\quad
\begin{align*}
\text{add} : & \quad a + b = c \\
\hline
\text{succ}(a) + b & = \text{succ}(c)
\end{align*}
\]

\textbf{data} \ Nat = \text{Zero} | \text{Succ} \ Nat

\textbf{deriving} (\text{Eq}, \text{Show})

\textbf{subNat} :: \ Nat \rightarrow \ Nat \rightarrow \text{Maybe} \ Nat

\begin{align*}
\text{subNat} b \text{ Zero} & = \text{Just} b \\
\text{subNat} (\text{Succ} \ c) (\text{Succ} \ a) & = \text{subNat} c a \\
\text{subNat} \text{ Zero} (\text{Succ} \ _) & = \text{Nothing}
\end{align*}

\begin{itemize}
\item -- Given \( c \) and \( a \), what \( b \) satisfies \( a + b = c \)?
\item -- addzero rule
\item -- add rule
\item -- failure
\end{itemize}

Still straightforward dataflow, but the function is no longer total
A Definition of Binary Trees

\[
\text{leaf} \quad \text{tree} \\
\text{branch}(t_1, t_2) \quad \text{tree}
\]

Judgments: \( t \) tree “\( t \) is a tree”
Variables: \( t_1 \quad t_2 \quad t_1 \) and \( t_2 \) may be equal
Symbols: \( \text{leaf} \quad \text{branch}(\ , \ ) \)
A Definition of Binary Trees

Judgments:  \( t \) tree  “\( t \) is a tree”
Variables: \( t_1 \quad t_2 \quad t_1 \) and \( t_2 \) may be equal
Symbols:  \( \text{leaf} \quad \text{branch}(\ , \ ) \)

Derivations are generally tree-structured

\[
\text{branch(\text{branch(\text{leaf,leaf}),leaf}),leaf)} \text{ tree}
\]
A Definition of Binary Trees

Judgments: \( t \text{ tree} \quad \text{“} t \text{ is a tree} \text{”} \)

Variables: \( t_1 \quad t_2 \quad t_1 \text{ and } t_2 \text{ may be equal} \)

Symbols: \( \text{leaf} \quad \text{branch}(\quad,\quad) \)

Derivations are generally tree-structured
A Definition of Binary Trees

Judgments: \( t \) tree “\( t \) is a tree”
Variables: \( t_1 \), \( t_2 \), \( t_1 \) and \( t_2 \) may be equal
Symbols: \( \text{leaf} \), \( \text{branch}(\ , \ ) \)

Derivations are generally tree-structured

\[
\begin{aligned}
\text{leaf} \quad \text{tree} \\
\text{branch}(\text{leaf},\text{leaf}) \quad \text{tree} \\
\text{leaf} \\
\text{branch}(\text{branch}(\text{leaf},\text{leaf}),\text{leaf}) \quad \text{tree}
\end{aligned}
\]
A Definition of Binary Trees

\[
\text{leaf} \quad \text{tree} \quad t_1 \quad \text{tree} \quad t_2 \quad \text{tree} \\
\text{branch}(t_1, t_2) \quad \text{tree}
\]

Judgments: \( t \) tree “\( t \) is a tree”

Variables: \( t_1, t_2 \) \( t_1 \) and \( t_2 \) may be equal

Symbols: leaf \, branch( , )

Derivations are generally tree-structured

\[
\text{leaf} \quad \text{tree} \quad \text{leaf} \quad \text{tree} \quad \text{leaf} \\
\text{leaf} \quad \text{tree} \quad \text{leaf} \quad \text{tree} \quad \text{leaf} \quad \text{tree} \\
\text{branch(leaf,leaf)} \quad \text{tree} \quad \text{branch(leaf,leaf),leaf) tree}
\]
A Definition of Binary Trees

Judgments: \( t \) tree “\( t \) is a tree”

Variables: \( t_1, t_2 \) \( t_1 \) and \( t_2 \) may be equal

Symbols: leaf \( \text{branch}(\ , \, \) \)

\[
\text{data Tree = Leaf | Branch Tree Tree}
\]

\[
isTree :: \text{Tree} \rightarrow \text{Bool}
\]

\[
isTree \text{ Leaf} = \text{True}
\]

\[
isTree \text{ (Branch l r)} = \text{isTree l \&\& isTree r} \quad -- \text{Must test both branches}
\]

Trivially true because of Haskell’s types, but note two-way recursion