Judgments, Inference Rules, and Inductive Definitions

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Spring 2023
There are various symbols

Symbols may be identical, even when drawn slightly differently. Other symbols are distinct.

Symbols arranged in a horizontal sequence are "words," "strings," or "expressions."

Symbols may represent values, operations, or relationships.

Some symbols are treated as variables that represent other symbols. The meaning of an expression with variables depends on the variables' values.
There are various symbols
Symbols may be identical, even when drawn slightly differently
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Other symbols are distinct
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Symbols arranged in a horizontal sequence are “words,” “strings,” or “expressions”
Symbols may represent values, operations, or relationships
Some symbols are treated as variables that represent other symbols

LETTERS CALLED VARIABLES HAVE VALUES THAT CAN CHANGE
2 + X
There are various symbols
Symbols may be identical, even when drawn slightly differently
Other symbols are distinct
Symbols arranged in a horizontal sequence are “words,” “strings,” or “expressions”
Symbols may represent values, operations, or relationships
Some symbols are treated as variables that represent other symbols
The meaning of an expression with variables depends on the variables’ values
A *judgment* is an assertion about one or more things, typically membership in a set.

- \(0 \in \mathbb{N}\): 0 is a member of the set of natural numbers
- \(\mathit{n \ nat}\): \(\mathit{n}\) is a member of the set of natural numbers
- \(\mathit{1 + 2 \ expr}\): \(\mathit{1 + 2}\) is in the set of expressions
- \(\mathit{\tau \ type}\): \(\mathit{\tau}\) is in the set of types
- \(\mathit{e : \tau}\): Expression \(\mathit{e}\) has type \(\mathit{\tau}\)
- \(\mathit{\text{sum}(n_1, n_2, n_3)}\): Adding \(n_1\) and \(n_2\) gives \(n_3\)
- \(\mathit{n_1 + n_2 = n_3}\): Adding \(n_1\) and \(n_2\) gives \(n_3\)

Prefix; infix; and suffix syntax
Inference Rule

Premises: Judgments → $J_1 \ J_2 \ \ldots \ J_k$
Conclusion: A Judgment → $J$

“If all the premises hold, the conclusion follows”

The Natural Numbers Defined Inductively by Two Inference Rules (Peano)

Axiom → $\emptyset \in \mathbb{N}$ zero

Judgments: $a \in \mathbb{N}$

Variables: $a$ ← Sequences of symbols

Symbols: $\emptyset$ succ( )

<table>
<thead>
<tr>
<th>0</th>
<th>0</th>
</tr>
</thead>
<tbody>
<tr>
<td>succ(0)</td>
<td>1</td>
</tr>
<tr>
<td>succ(succ(0))</td>
<td>2</td>
</tr>
<tr>
<td>succ(succ(succ(0)))</td>
<td>3</td>
</tr>
<tr>
<td>succ(succ(succ(succ(0))))</td>
<td>4</td>
</tr>
</tbody>
</table>
Inference Rule

Premises: Judgments $\rightarrow J_1 J_2 \ldots J_k$
Conclusion: A Judgment $\rightarrow J$

“If all the premises hold, the conclusion follows”

The Natural Numbers Defined Inductively by Two Inference Rules (Peano)

$0 \in \mathbb{N}$ zero

$\exists a \in \mathbb{N}$ successor

Judgments: $a \in \mathbb{N}$
Variables: $a \leftarrow$ Sequences of symbols
Symbols: $0 \quad \text{succ}(\quad)$

<table>
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<td>3</td>
</tr>
<tr>
<td>succ(succ(succ(succ(0))))</td>
<td>4</td>
</tr>
</tbody>
</table>
Inference Rule

Premises: Judgments → $\vdash J_1 \quad J_2 \quad \ldots \quad J_k$
Conclusion: A Judgment → $\vdash J$

“If all the premises hold, the conclusion follows”

The Natural Numbers Defined Inductively by Two Inference Rules (Peano)

$\vdash 0 \in \mathbb{N}$ (zero)
$\vdash a \in \mathbb{N} \quad \Rightarrow \quad \text{successor}$ (successor)

Technically a scheme

Scheme: pattern with variables: replacing $a$ consistently gives a rule

$\vdash 0 \in \mathbb{N}$
$\vdash \text{succ}(0) \in \mathbb{N}$
$\vdash \text{true} \in \mathbb{N}$
$\vdash \text{succ}(\text{succ}(0)) \in \mathbb{N}$
$\vdash \text{succ}($true$) \in \mathbb{N}$

Which are variables? Values constrained? Variable scope: a single rule

Consistent replacement only:

$\vdash \text{foo} \in \mathbb{N}$

is not a rule
Inference Rule

Premises: Judgments \( \rightarrow \) \( J_1 \) \( J_2 \) \( \cdots \) \( J_k \) \Rule-Name

Conclusion: A Judgment \( \rightarrow \) \( J \)

“If all the premises hold, the conclusion follows”

The Natural Numbers Defined Inductively by Two Inference Rules (Peano)

\[
\begin{align*}
0 & \in \mathbb{N} \quad \text{zero} \\
\text{succ}(a) & \in \mathbb{N} \quad \text{successor}
\end{align*}
\]

Is \( \text{succ}(\text{succ}(\text{succ}(0))) \) a.k.a. 3 a natural number? A forward derivation

\[
\begin{align*}
0 & \in \mathbb{N} \quad \text{zero}
\end{align*}
\]
Inference Rule

Premises: Judgments → $J_1, J_2, \ldots, J_k$ → Conclusion: A Judgment → $J$

“If all the premises hold, the conclusion follows”

The Natural Numbers Defined Inductively by Two Inference Rules (Peano)

- $\emptyset \in \mathbb{N}$ zero
- $a \in \mathbb{N}$ successor

Is $\text{succ}(%28%28%28%5Ctext{succ}(%5Cemptyset)%29%29%29)$ a.k.a. 3 a natural number? A forward derivation

- $\emptyset \in \mathbb{N}$ zero
- $\text{succ}(\emptyset) \in \mathbb{N}$ successor ← choose $a = \emptyset$
Inference Rule

Premises: Judgments → \( J_1, J_2, \ldots, J_k \)  
Conclusion: A Judgment → \( \frac{J}{J} \)  

“If all the premises hold, the conclusion follows”

The Natural Numbers Defined Inductively by Two Inference Rules (Peano)

0 ∈ \( \mathbb{N} \)  
\( a \in \mathbb{N} \)  
\( \text{succ}(a) \in \mathbb{N} \)

Is \( \text{succ}(\text{succ}(\text{succ}(0))) \) a.k.a. 3 a natural number? A forward derivation

0 ∈ \( \mathbb{N} \)  
\( \text{succ}(0) \in \mathbb{N} \)  
\( \text{succ}(\text{succ}(0)) \in \mathbb{N} \)  
\( \text{choose} \ a = \text{succ}(0) \)
Inference Rule

Premises: Judgments \( \rightarrow \) \( J_1 \ J_2 \ \cdots \ J_k \) \( \vdash \) \( J \)  
Conclusion: A Judgment \( \rightarrow \) 

"If all the premises hold, the conclusion follows"

The Natural Numbers Defined Inductively by Two Inference Rules (Peano)

\[
\begin{align*}
\text{zero} & : 0 \in \mathbb{N} \\
\text{successor} & : a \in \mathbb{N} \implies \text{succ}(a) \in \mathbb{N}
\end{align*}
\]

Is \( \text{succ}(\text{succ}(\text{succ}(0))) \) a.k.a. 3 a natural number? A forward derivation

\[
\begin{align*}
\text{zero} & : 0 \in \mathbb{N} \\
\text{successor} & : \text{succ}(0) \in \mathbb{N} \\
\text{successor} & : \text{succ}(\text{succ}(0)) \in \mathbb{N} \\
\text{successor} & : \text{succ}(\text{succ}(\text{succ}(0))) \in \mathbb{N}
\end{align*}
\]
The Natural Numbers

\[
\begin{align*}
\text{zero} & \quad \in \mathbb{N} \\
0 & \in \mathbb{N} \\
\text{successor} & \quad a \in \mathbb{N} \\
\text{succ}(a) & \in \mathbb{N}
\end{align*}
\]

\[
\begin{align*}
\text{zeroIsN} & \quad :: \text{String} \rightarrow \text{Bool} \\
\text{successorIsN} & \quad :: \text{String} \rightarrow \text{Bool}
\end{align*}
\]

String is inefficient, but let’s focus on correctness first
The Natural Numbers

\[
\begin{align*}
\text{zero} & \quad \in \mathbb{N} \\
0 & \in \mathbb{N} \\
\text{successor} & \quad \in \mathbb{N} \\
\text{succ}(a) & \in \mathbb{N}
\end{align*}
\]

\[
\begin{align*}
\text{zeroIsN} & : \text{String} \to \text{Bool} \\
\text{zeroIsN} \ "0" & = \text{True} \\
\text{zeroIsN} \ _ & = \text{False} \quad -- \text{Default case}
\end{align*}
\]

\[
\begin{align*}
\text{successorIsN} & : \text{String} \to \text{Bool}
\end{align*}
\]
The Natural Numbers

\[ \begin{align*}
\text{zero} & \in \mathbb{N} \\
\text{successor} & \in \mathbb{N}
\end{align*} \]

```haskell
import Data.List (stripPrefix) -- stripPrefix :: String -> String -> Maybe String

zeroIsN :: String -> Bool
zeroIsN "0" = True
zeroIsN _  = False  -- Default case

successorIsN :: String -> Bool  -- Construct a Reverse Derivation
successorIsN s = case stripPrefix "succ(" s of
  Just aa@(_:_) -> last aa == ')' &&
    let a = init aa in
      zeroIsN a || successorIsN a  -- Try both
  _                   -> False
```

---

Note: The above code snippet is a simplified representation and may not correctly handle all edge cases due to the limitations of the plain text format.
The Natural Numbers

\[
\begin{align*}
\text{zero} & : 0 \in \mathbb{N} \\
\text{successor} & : \text{succ}(a) \in \mathbb{N}
\end{align*}
\]

import Data.List (stripPrefix)

zeroIsN :: String -> Bool
zeroIsN "0" = True
zeroIsN _ = False

successorIsN :: String -> Bool
successorIsN s = case match "succ(" s of
  Just a -> zeroIsN a || successorIsN a
_ -> False

match :: String -> String -> String -> Maybe String -- Helper function
match pre suff s = do a' <- stripPrefix pre s
                    reverse <$> stripPrefix (reverse suff) (reverse a')
The Natural Numbers

\[
\begin{align*}
0 & \in \mathbb{N} \\
\text{succ}(a) & \in \mathbb{N}
\end{align*}
\]

\[a \in \mathbb{N}\]

---

import Data.List (stripPrefix)

isNat :: String -> Bool
isNat "0" = True
isNat s  = case match "succ(" ")" s of
    Just a -> isNat a
    Nothing -> False

match :: String -> String -> String -> Maybe String
match pre suff s = do a' <- stripPrefix pre s
                      reverse <$> stripPrefix (reverse suff) (reverse a')
The Natural Numbers

\[
\begin{align*}
0 & \in \mathbb{N} \\
\text{succ}(a) & \in \mathbb{N}
\end{align*}
\]

\[
\begin{align*}
\text{data} \quad \text{Nat} &= \text{Zero} \mid \text{Succ Nat} & \text{-- Algebraic data type: either “Zero” or “Succ n”} \\
\text{zeroIsN} & \quad :: \text{Nat} \rightarrow \text{Bool} \\
\text{zeroIsN Zero} &= \text{True} \\
\text{zeroIsN }_\_ &= \text{False} \\
\text{successorIsN} & \quad :: \text{Nat} \rightarrow \text{Bool} \\
\text{successorIsN (Succ a)} &= \text{zeroIsN a} \mid\mid \text{successorIsN a} & \text{-- Try both} \\
\text{successorIsN }_\_ &= \text{False}
\end{align*}
\]
The Natural Numbers

\[ \begin{align*}
\text{zero} & \quad 0 \in \mathbb{N} \\
\text{successor} & \quad a \in \mathbb{N} \quad \Rightarrow \quad \text{succ}(a) \in \mathbb{N}
\end{align*} \]

\textbf{data} Nat = Zero | Succ Nat

\text{isNat} :: Nat -> \text{Bool}
\text{isNat Zero} = \textbf{True} \quad -- \text{zero rule}
\text{isNat (Succ a)} = \text{isNat a} \quad -- \text{successor rule}

isNat is trivial; Haskell’s type system enforces it for us
Equality of Natural Numbers as an Inductive Definition

\[
\begin{array}{c}
\text{equalzero} \\
\text{eq}(0,0)
\end{array}
\quad
\begin{array}{c}
equal \\
eq(a,b)
\end{array}
\quad
\begin{array}{c}
equal \\
eq(\text{succ}(a),\text{succ}(b))
\end{array}
\]

Judgements: \(\text{eq}(n_1,n_2)\) “\(n_1\) and \(n_2\) are equal” ← a relation/a set of pairs

Variables: \(a\quad b\)

Symbols: \(\emptyset\quad\text{succ}(\quad)\)

Is 3 = 3? A reverse derivation

\[
\text{eq}(\text{succ}(\text{succ}(\text{succ}(\emptyset))),\text{succ}(\text{succ}(\text{succ}(\emptyset))))
\]
Equality of Natural Numbers as an Inductive Definition

\[
\begin{align*}
\text{equalzero} & : \text{eq}(0, 0) \\
\text{equal} & : \text{eq}(\text{succ}(a), \text{succ}(b)) \Rightarrow \text{eq}(a, b)
\end{align*}
\]

Judgements: \( \text{eq}(n_1, n_2) \) “\( n_1 \) and \( n_2 \) are equal”

Variables: \( a \quad b \)

Symbols: \( 0 \quad \text{succ}(\quad) \)

Is 3 = 3? A reverse derivation

\[
\begin{align*}
\text{eq}(\text{succ}(\text{succ}(\text{succ}(0))), \text{succ}(\text{succ}(\text{succ}(0)))) & \Rightarrow \\
\text{eq}(\text{succ}(\text{succ}(0)), \text{succ}(\text{succ}(0))) & \Rightarrow \\
\text{eq}(0, 0) & \Rightarrow \\
\text{eq}(\text{succ}(0), \text{succ}(0)) & \Rightarrow \\
\text{eq}(0, 0) & \Rightarrow \\
\text{eq}(\text{succ}(0), \text{succ}(0)) & \Rightarrow \\
\text{eq}(0, 0) & \Rightarrow
\end{align*}
\]
Equality of Natural Numbers as an Inductive Definition

\[
\begin{align*}
\text{equalzero} & : \text{eq}(0, 0) \\
\text{equal} & : \text{eq}(\text{succ}(a), \text{succ}(b)) \\
\end{align*}
\]

Judgements: \( \text{eq}(n_1, n_2) \) “\( n_1 \) and \( n_2 \) are equal”

Variables: \( a \quad b \)

Symbols: \( 0 \quad \text{succ}(\quad) \)

Is \( 3 = 3 \)? A reverse derivation

\[
\begin{align*}
\text{eq}(\text{succ}(0), \text{succ}(0)) \quad \text{eq}(\text{succ}(\text{succ}(0)), \text{succ}(\text{succ}(0))) \quad \text{eq}(\text{succ}(\text{succ}(\text{succ}(0))), \text{succ}(\text{succ}(\text{succ}(0))))
\end{align*}
\]
Equality of Natural Numbers as an Inductive Definition

\[
\begin{align*}
\text{equalzero:} & \quad \text{eq}(0, 0) \\
\text{equal:} & \quad \frac{\text{eq}(a, b)}{\text{eq}(\text{succ}(a), \text{succ}(b))}
\end{align*}
\]

Judgements: \( \text{eq}(n_1, n_2) \) “\( n_1 \) and \( n_2 \) are equal”

Variables: \( a \quad b \)

Symbols: \( 0 \quad \text{succ}(\quad) \)

Is 3 = 3? A reverse derivation

\[
\begin{align*}
\text{eq}(0, 0) & \quad \text{equalzero} \\
\text{eq}(\text{succ}(0), \text{succ}(0)) & \quad \text{eq}(0, 0) \\
\text{eq}(\text{succ}(\text{succ}(0)), \text{succ}(\text{succ}(0))) & \quad \text{equal} \\
\text{eq}(\text{succ}(\text{succ}(\text{succ}(0))), \text{succ}(\text{succ}(\text{succ}(0)))) & \quad \text{equal}
\end{align*}
\]

Is 1 = 2?

\[
\begin{align*}
\text{eq}(0, \text{succ}(0)) & \quad \text{equalzero} \\
\text{eq}(\text{succ}(0), \text{succ}(\text{succ}(0))) & \quad \text{eq}(0, \text{succ}(0)) \\
\text{eq}(\text{succ}(\text{succ}(0)), \text{succ}(\text{succ}(\text{succ}(0)))) & \quad \text{equal}
\end{align*}
\]

We are stuck: neither rule applies, so \( 1 \neq 2 \)
Equality of Natural Numbers as an Inductive Definition

\[
\begin{align*}
\text{eq}(0, 0) & \quad \text{equalzero} \\
\text{eq}(\text{succ}(a), \text{succ}(b)) & \quad \text{equal} \\
\end{align*}
\]

Judgements: \( \text{eq}(n_1, n_2) \) “\( n_1 \) and \( n_2 \) are equal”

Variables: \( a \quad b \)

Symbols: \( 0 \quad \text{succ}( ) \)

Is 3 = 3? A reverse derivation

\[
\begin{align*}
\text{eq}(\text{succ}(\text{succ}(\text{succ}(0))), \text{succ}(\text{succ}(\text{succ}(0)))) & \quad \text{equal} \\
\text{eq}(\text{succ}(\text{succ}(0)), \text{succ}(\text{succ}(0))) & \quad \text{equal} \\
\text{eq}(\text{succ}(0), \text{succ}(0)) & \quad \text{equal} \\
\text{eq}(0, 0) & \quad \text{equalzero}
\end{align*}
\]

We are stuck: neither rule applies, so 1 \( \neq \) 2
Equality of Natural Numbers as an Inductive Definition

\[
\begin{align*}
\text{eq}(0, 0) & \quad \text{equalzero} \\
\text{eq}(\text{succ}(a), \text{succ}(b)) & \quad \text{equal}
\end{align*}
\]

Judgements: \( \text{eq}(n_1, n_2) \) “\( n_1 \) and \( n_2 \) are equal”

Variables: \( a \quad b \)

Symbols: \( 0 \quad \text{succ}(\quad) \)

Is 3 = 3? A reverse derivation

\[
\begin{align*}
\text{eq}(0, 0) & \quad \text{equalzero} \\
\text{eq}(\text{succ}(0), \text{succ}(0)) & \quad \text{equal} \\
\text{eq}(\text{succ}(\text{succ}(0)), \text{succ}(\text{succ}(0))) & \quad \text{equal} \\
\text{eq}(\text{succ}(\text{succ}(\text{succ}(0))), \text{succ}(\text{succ}(\text{succ}(0)))) & \quad \text{equal}
\end{align*}
\]

Is 1 = 2?

\( \text{eq}(\text{succ}(0), \text{succ}(\text{succ}(0))) \)
Equality of Natural Numbers as an Inductive Definition

\[
\begin{align*}
\text{equalzero:} & \quad \text{eq}(0, 0) \\
\text{equal:} & \quad \text{eq}(\text{succ}(a), \text{succ}(b))
\end{align*}
\]

Judgements: \( \text{eq}(n_1, n_2) \) “\( n_1 \) and \( n_2 \) are equal”
Variables: \( a \quad b \)
Symbols: \( 0 \quad \text{succ}(\quad) \)

Is 3 = 3? A reverse derivation

\[
\begin{align*}
\text{eq}(0, 0) & \quad \text{equalzero} \\
\text{eq}(\text{succ}(0), \text{succ}(0)) & \quad \text{equal} \\
\text{eq}(\text{succ}(\text{succ}(0)), \text{succ}(\text{succ}(0))) & \quad \text{equal} \\
\text{eq}(\text{succ}(\text{succ}(\text{succ}(0))), \text{succ}(\text{succ}(\text{succ}(0)))) & \quad \text{equal}
\end{align*}
\]

Is 1 = 2?

\[
\begin{align*}
\text{eq}(0, \text{succ}(0)) & \quad \text{equal} \\
\text{eq}(\text{succ}(0), \text{succ}(\text{succ}(0))) & \quad \text{equal}
\end{align*}
\]
Equality of Natural Numbers as an Inductive Definition

\[ \begin{align*}
\text{eq}(0, 0) & \quad \text{equalzero} \\
\text{eq}(\text{succ}(a), \text{succ}(b)) & \quad \text{equal}
\end{align*}\]

Judgements: \( \text{eq}(n_1, n_2) \) “\( n_1 \) and \( n_2 \) are equal”

Variables: \( a \quad b \)

Symbols: \( 0 \quad \text{succ}(\quad) \)

Is \( 3 = 3? \) A reverse derivation

\[ \begin{align*}
\text{eq}(0, 0) & \quad \text{equalzero} \\
\text{eq}(\text{succ}(0), \text{succ}(0)) & \quad \text{equal} \\
\text{eq}(\text{succ}(\text{succ}(0)), \text{succ}(\text{succ}(0))) & \quad \text{equal} \\
\text{eq}(\text{succ}(\text{succ}(\text{succ}(0))), \text{succ}(\text{succ}(\text{succ}(0)))) & \quad \text{equal}
\end{align*}\]

Is \( 1 = 2? \)

\[ \begin{align*}
\text{eq}(0, \text{succ}(0)) & \quad \text{equal} \\
\text{eq}(\text{succ}(0), \text{succ}(\text{succ}(0))) & \quad \text{equal}
\end{align*}\]

We are stuck: neither rule applies, so \( 1 \neq 2 \)
Equality of Natural Numbers as an Inductive Definition

\[
\begin{align*}
0 &= 0 \quad \text{equalzero} \\
\text{succ}(a) &= \text{succ}(b) \quad \text{equal}
\end{align*}
\]

Judgements: \( n_1 = n_2 \) “\( n_1 \) and \( n_2 \) are equal” ← a relation/a set of pairs

Variables: \( a \quad b \)

Symbols: \( 0 \quad \text{succ}(\quad) \)

Is 3 = 3?

\[
\begin{align*}
0 &= 0 \quad \text{equalzero} \\
\text{succ}(0) &= \text{succ}(0) \quad \text{equal} \\
\text{succ}(\text{succ}(0)) &= \text{succ}(\text{succ}(0)) \quad \text{equal} \\
\text{succ}(\text{succ}(\text{succ}(0))) &= \text{succ}(\text{succ}(\text{succ}(0))) \quad \text{equal}
\end{align*}
\]

Is 1 = 2?

\[
\begin{align*}
0 &= \text{succ}(0) \quad \text{equal} \\
\text{succ}(0) &= \text{succ}(\text{succ}(0))
\end{align*}
\]

We are stuck: neither rule applies, so \( 1 \neq 2 \)
Equality of Natural Numbers as an Inductive Definition

\[
\begin{align*}
\frac{0 = 0}{\text{equalzero}} & \quad \frac{a = b}{\text{equal}} \\
\text{data} & \quad \text{Nat} = \text{Zero} \mid \text{Succ Nat} \\
\text{natEqual} :: & \quad \text{Nat} \to \text{Nat} \to \text{Bool} \\
\text{natEqual Zero Zero} & = \text{True} \quad \text{-- equalzero rule} \\
\text{natEqual (Succ a) (Succ b)} & = \text{natEqual a b} \quad \text{-- equal rule} \\
\text{natEqual _ _} & = \text{False} \\
\text{Again: single function because only one rule may ever match}
\end{align*}
\]
Equality of Natural Numbers as an Inductive Definition

\[
\begin{align*}
\emptyset = \emptyset & \quad \text{equalzero} \\
\text{succ}(a) = \text{succ}(b) & \quad \text{equal}
\end{align*}
\]

data Nat = Zero | Succ Nat
deriving Eq

This Haskell’s default implementation of == for algebraic data types
Addition as an Inductive Definition

\[
\begin{align*}
  b \in \mathbb{N} & \quad \text{addzero} \\
  \text{sum}(\emptyset, b, b) & \\
  \frac{\text{sum}(a, b, c)}{\text{add}} & \frac{\text{sum}(\text{succ}(a), b, \text{succ}(c))}
\end{align*}
\]

Judgments: \( n \in \mathbb{N} \quad \text{sum}(n_1, n_2, n_3) \leftarrow \text{a relation/a set of triples} \)

Variables: \( a \quad b \quad c \)

Symbols: \( \emptyset \quad \text{succ}(\quad) \)
Addition as an Inductive Definition

\[
\begin{align*}
& b \in \mathbb{N} \\
\frac{\text{addzero}}{\text{sum}(\emptyset, b, b)} \\
& \frac{\text{add}}{\text{sum}(\text{succ}(a), b, \text{succ}(c))}
\end{align*}
\]

Judgments: \( n \in \mathbb{N} \quad \text{sum}(n_1, n_2, n_3) \)

Variables: \( a \quad b \quad c \)

Symbols: \( 0 \quad \text{succ}() \)

Is \( 2 + 1 = 3? \)

\[
\text{sum}(
\text{succ}(
\text{succ}(\emptyset)
),
\text{succ}(\emptyset),
\text{succ}(
\text{succ}(
\text{succ}(
\text{succ}(\emptyset)
))))
\]
Addition as an Inductive Definition

\[
\begin{align*}
  b \in \mathbb{N} & \quad \text{addzero} \\
  \text{sum}(0, b, b) & \\
  \text{sum}(a, b, c) & \quad \text{add} \\
  \text{sum}(\text{succ}(a), b, \text{succ}(c)) & \\
\end{align*}
\]

Judgments: \( n \in \mathbb{N} \quad \text{sum}(n_1, n_2, n_3) \)

Variables: \( a \quad b \quad c \)

Symbols: \( 0 \quad \text{succ}() \)

Is \( 2 + 1 = 3? \)

\[
\begin{align*}
  \text{sum}(\text{succ}(0), \text{succ}(0), \text{succ}(\text{succ}(0))) & \quad \text{add} \\
  \text{sum}(\text{succ}(\text{succ}(0)), \text{succ}(0), \text{succ}(\text{succ}(\text{succ}(0)))) & \\
\end{align*}
\]
Addition as an Inductive Definition

\[
\begin{align*}
  b \in \mathbb{N} & \quad \frac{\text{sum}(\emptyset, b, b)}{\text{addzero}} \\
  \text{sum}(a, b, c) & \quad \frac{\text{sum}(\text{succ}(a), b, \text{succ}(c))}{\text{add}}
\end{align*}
\]

Judgments: \( n \in \mathbb{N} \quad \text{sum}(n_1, n_2, n_3) \)

Variables: \( a \quad b \quad c \)

Symbols: \( \emptyset \quad \text{succ}(\quad) \)

Is \( 2 + 1 = 3 \)?

\[
\begin{align*}
  \text{sum}(\emptyset, \text{succ}(\emptyset), \text{succ}(\emptyset)) & \quad \frac{\text{add}}{\text{add}} \\
  \text{sum}(\text{succ}(\emptyset), \text{succ}(\emptyset), \text{succ}(\text{succ}(\emptyset))) & \quad \frac{\text{add}}{\text{add}} \\
  \text{sum}(& \text{succ}(\text{succ}(\emptyset)), \text{succ}(\emptyset), \text{succ}(\text{succ}(\text{succ}(\emptyset)))) \quad \frac{\text{add}}{\text{add}}
\end{align*}
\]
Addition as an Inductive Definition

\[
\begin{align*}
\text{addzero:} & \quad b \in \mathbb{N} \quad \frac{\text{sum}(0, b, b)}{\text{sum}(\text{succ}(a), b, \text{succ}(c))} \\
\text{add:} & \quad \frac{\text{sum}(a, b, c)}{\text{sum}(\text{succ}(a), b, \text{succ}(c))}
\end{align*}
\]

Judgments: \( n \in \mathbb{N} \quad \text{sum}(n_1, n_2, n_3) \)

Variables: \( a \quad b \quad c \)

Symbols: \( 0 \quad \text{succ}(\quad) \)

Is \( 2 + 1 = 3? \)

\[
\begin{align*}
\text{addzero:} & \quad \text{succ}(0) \in \mathbb{N} \quad \frac{\text{sum}(0, \text{succ}(0), \text{succ}(0))}{\text{sum}(\text{succ}(0), \text{succ}(0), \text{succ}(\text{succ}(0))))} \\
\text{add:} & \quad \frac{\text{sum}(\text{succ}(0), \text{succ}(0), \text{succ}(\text{succ}(0))))}{\text{sum}(\text{succ}(\text{succ}(0))), \text{succ}(0), \text{succ}(\text{succ}(\text{succ}(0))))}
\end{align*}
\]
Addition as an Inductive Definition

\[
\begin{align*}
b \in \mathbb{N} & \quad \frac{\text{sum}(\emptyset, b, b)}{\text{addzero}} \quad \frac{\text{sum}(a, b, c)}{\text{add}} \\
\text{sum}(\text{succ}(a), b, \text{succ}(c)) & \end{align*}
\]

Judgments: \( n \in \mathbb{N} \quad \text{sum}(n_1, n_2, n_3) \)

Variables: \( a \ b \ c \)

Symbols: \( 0 \ \text{succ}(\ ) \)

Is \( 2 + 1 = 3? \)

\[
\begin{align*}
0 \in \mathbb{N} & \quad \frac{\text{succ}(0) \in \mathbb{N}}{\text{successor}} \\
\text{sum}(0, \text{succ}(0), \text{succ}(0)) & \quad \frac{\text{addzero}}{\text{add}} \\
\text{sum}(\text{succ}(0), \text{succ}(0), \text{succ}(\text{succ}(0))) & \quad \frac{\text{add}}{\text{add}} \\
\text{sum}(\text{succ}(\text{succ}(0)), \text{succ}(0), \text{succ}(\text{succ}(\text{succ}(0)))) & \end{align*}
\]
Addition as an Inductive Definition

\[
\frac{b \in \mathbb{N}}{\text{sum}(\theta, b, b)} \quad \text{addzero} \quad \frac{\text{sum}(a, b, c)}{\text{sum}(\text{succ}(a), b, \text{succ}(c))} \quad \text{add}
\]

Judgments: \( n \in \mathbb{N} \quad \text{sum}(n_1, n_2, n_3) \)

Variables: \( a \quad b \quad c \)

Symbols: \( \theta \quad \text{succ}(\phantom{0}) \)

Is \( 2 + 1 = 3 \)?

\[
\frac{\theta \in \mathbb{N}}{\text{successor}} \quad \frac{\text{successor}}{\text{successor}} \quad \frac{\text{addzero}}{\text{add}} \quad \frac{\text{add}}{\text{add}} \quad \frac{\text{add}}{\text{add}} \quad \frac{\text{add}}{\text{add}}
\]

\[
\text{sum}(\theta, \text{succ}(\theta), \text{succ}(\theta)) \quad \text{sum}(\text{succ}(\theta), \text{succ}(\theta), \text{succ}(\text{succ}(\theta))) \quad \text{sum}(\text{succ}(\text{succ}(\theta)), \text{succ}(\theta), \text{succ}(\text{succ}(\text{succ}(\theta))))
\]
Addition as an Inductive Definition

\[
\begin{align*}
\text{addzero} : & \quad \emptyset + b = b \\
\text{add} : & \quad a + b = c \\
\end{align*}
\]

Judgments: \( n \in \mathbb{N} \quad n_1 + n_2 = n_3 \) ← a relation/a set of triples

Variables: \( a \quad b \quad c \)

Symbols: \( 0 \quad \text{succ}(\quad) \)

Is \( 2 + 1 = 3? \)

\[
\begin{align*}
\text{zero} : & \quad \emptyset \in \mathbb{N} \\
\text{successor} : & \quad \text{succ}(\emptyset) \in \mathbb{N} \\
\text{addzero} : & \quad \emptyset + \text{succ}(\emptyset) = \text{succ}(\emptyset) \\
\text{add} : & \quad \text{succ}(\emptyset) + \text{succ}(\emptyset) = \text{succ}(\text{succ}(\emptyset)) \\
\text{add} : & \quad \text{succ}(\text{succ}(\emptyset)) + \text{succ}(\emptyset) = \text{succ}(\text{succ}(\text{succ}(\emptyset)))
\end{align*}
\]
Addition as an Inductive Definition

\[
  \frac{b \in \mathbb{N}}{\emptyset + b = b} \quad \text{addzero} \quad \frac{a + b = c}{\text{succ}(a) + b = \text{succ}(c)} \quad \text{add}
\]

\[
\begin{align*}
\text{data Nat} & = \text{Zero} \mid \text{Succ Nat} \\
\text{deriving Eq}
\end{align*}
\]

\[
\begin{align*}
\text{sumsTo} :: \text{Nat} \to \text{Nat} \to \text{Nat} \to \text{Bool} & \quad \text{-- Is } a + b = c? \\
\text{sumsTo Zero } b \ b' & \mid b == b' = \text{True} \quad \text{-- addzero rule} \\
\text{sumsTo } \text{Succ } a \ b \ (\text{Succ } c) & = \text{sumsTo } a \ b \ c \quad \text{-- add rule} \\
\text{sumsTo } _ & _ _ = \text{False} \quad \text{-- E.g., } (\text{Succ } a) \ _ \ \text{Zero}
\end{align*}
\]

No need to check whether \( b \in \mathbb{N} \): the types enforce this

Haskell patterns can’t check for equality like \text{sumsTo Zero } b \ b \), so I added guard \( b == 'b \)

Rather awkward to ask “is this it?”
Addition as an Inductive Definition

\[
\begin{align*}
  b \in \mathbb{N} \\
  \theta + b &= b & \text{addzero} \\
  a + b &= c & \text{add} \\
  \text{succ}(a) + b &= \text{succ}(c)
\end{align*}
\]

\textbf{data} \text{ Nat } = \text{ Zero } \mid \text{ Succ Nat}

\textbf{deriving} \ (\text{Eq, Show})

\text{addNat :: Nat } \rightarrow \text{ Nat } \rightarrow \text{ Nat} \quad \text{-- Given } a \text{ and } b, \text{ what } c \text{ satisfies } a + b = c? \\
\text{addNat Zero } b = b \quad \text{-- addzero rule} \\
\text{addNat (Succ a) } b = \text{ Succ (addNat a } b) \quad \text{-- add rule}

The dataflow makes this easy and it’s obviously a total function
Addition as an Inductive Definition

\[
\begin{align*}
    b \in \mathbb{N} & \quad \text{addzero} \\
    \theta + b &= b \\
    a + b &= c & \frac{\text{add}}{
    \text{succ}(a) + b = \text{succ}(c)}
\end{align*}
\]

data Nat = Zero | Succ Nat
deriving (Eq, Show)

subNat :: Nat -> Nat -> \text{Maybe Nat}
subNat c Zero = Just c -- addzero rule
subNat (Succ c) (Succ a) = subNat c a -- add rule
subNat Zero (Succ _) = Nothing -- failure

Still straightforward dataflow, but the function is no longer total
A Definition of Binary Trees

Judgments: \( t \) tree  “\( t \) is a tree”
Variables: \( t_1 \) \( t_2 \) \( t_1 \) and \( t_2 \) may be equal
Symbols: \( \text{leaf} \) \( \text{branch}( , , ) \)
A Definition of Binary Trees

\[
\text{leaf tree} \quad \frac{t_1 \text{ tree} \quad t_2 \text{ tree}}{\text{branch}(t_1, t_2) \text{ tree}}
\]

Judgments:  \( t \) tree  “\( t \) is a tree”
Variables:  \( t_1 \quad t_2 \quad t_1 \) and \( t_2 \) may be equal
Symbols:  leaf  branch(  ,  )

Derivations are generally tree-structured

\[
\text{branch(} \text{branch(leaf,leaf),leaf) tree}
\]
A Definition of Binary Trees

Judgments: \( t \) tree “\( t \) is a tree”

Variables: \( t_1 \) \( t_2 \) \( t_1 \) and \( t_2 \) may be equal

Symbols: leaf \( \text{branch}() \)

Derivations are generally tree-structured
A Definition of Binary Trees

\[
\text{leaf} \quad \text{tree} \quad \text{branch} \\
\text{leaf} \quad \text{tree} \quad \text{branch} \quad \text{tree} \\
\text{branch(} t_1, t_2 \text{)} \quad \text{tree}
\]

Judgments:  
\( t \) tree  “\( t \) is a tree”

Variables:  
\( t_1 \quad t_2 \quad t_1 \) and \( t_2 \) may be equal

Symbols:  
\text{leaf} \quad \text{branch( } , )

Derivations are generally tree-structured

\[
\text{leaf} \quad \text{tree} \quad \text{branch} \quad \text{tree} \quad \text{branch} \\
\text{branch(leaf,leaf)} \quad \text{tree} \quad \text{leaf} \quad \text{tree} \quad \text{branch} \quad \text{branch(leaf,leaf),leaf) tree}
\]
A Definition of Binary Trees

Judgments: $t$ tree "$t$ is a tree"
Variables: $t_1$, $t_2$ $t_1$ and $t_2$ may be equal
Symbols: leaf branch($,$)

Derivations are generally tree-structured
A Definition of Binary Trees

Judgments: \( t \) tree  “\( t \) is a tree”

Variables: \( t_1, t_2 \)  \( t_1 \) and \( t_2 \) may be equal

Symbols: leaf  branch(, )

data Tree = Leaf | Branch Tree Tree

isTree :: Tree -> Bool
isTree Leaf = True
isTree (Branch l r) = isTree l && isTree r  -- Must test both branches

Trivially true because of Haskell’s types, but note two-way recursion