CSEE 3827: Fundamentals of Computer Systems

Lecture 4 & 5

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Martha Kim
martha@cs.columbia.edu

Standard forms (redux)

Product and sum terms

- Product term: logical AND of literals (e.g., $X\overline{Y}Z$)
- Sum term: logical OR of literals (e.g., $A + \overline{B} + C$)

Minterms

A	B	C	minterm
0	0	0	m0 $\bar{A}\bar{B}\bar{C}$
0	0	1	m1 $\bar{A}\bar{B}C$
0	1	0	m2 $\bar{A}B\bar{C}$
0	1	1	m3 $\bar{A}BC$
1	0	0	m4 $A\bar{B}\bar{C}$
1	0	1	m5 $A\bar{B}C$
1	1	0	m6 $AB\bar{C}$
1	1	1	m7 ABC

- A product term in which all variables appear once, either complemented or uncomplemented.
- Each minterm evaluates to 1 for exactly one variable assignment, 0 for all others.
- Denoted by mX where X corresponds to the variable assignment for which mX = 1.

Sum of minterms form

- The logical OR of all minterms for which F = 1.

A	B	C	minterm		F
0	0	0	m0	$\overline{A}\,\overline{B}\,\overline{C}$	0
0	0	1	m1	$\overline{A}\,\overline{B}C$	1
0	1	0	m2	$\overline{A}B\overline{C}$	1
0	1	1	m3	$\overline{A}BC$	1
1	0	0	m4	$A\overline{B}\,\overline{C}$	0
1	0	1	m5	$A\overline{B}C$	0
1	1	0	m6	$AB\overline{C}$	0
1	1	1	m7	ABC	0

$$F = \overline{A}\,\overline{B}C + \overline{A}B\overline{C} + \overline{A}BC$$

$$= m1 + m2 + m3$$

$$= \sum m(1,2,3)$$

Sum of minterms form (2)

- The logical OR of all minterms for which F = 1.

A	B	C	minterm		F
0	0	0	m0	$\bar{A}\bar{B}\bar{C}$	0
0	0	1	m1	$\bar{A}\bar{B}C$	1
0	1	0	m2	$\bar{A}B\bar{C}$	1
0	1	1	m3	$\bar{A}BC$	1
1	0	0	m4	$A\bar{B}\bar{C}$	0
1	0	1	m5	$A\bar{B}C$	0
1	1	0	m6	$AB\bar{C}$	0
1	1	1	m7	ABC	0

m0	m1		m2		m3	m4	m5	m6	m7
1	0	+	0	+	0	0	0	0	0
0	1	+	0	+	0	0	0	0	0
0	0	+	1	+	0	0	0	0	0
0	0	+	0	+	1	0	0	0	0
0	0	+	0	+	0	1	0	0	0
0	0	+	0	+	0	0	1	0	0
0	0	+	0	+	0	0	0	1	0
0	0	+	0	+	0	0	0	0	1

Sum of minterms form (3)

- What is $\overline{F}$ in sum of minterms form?

A	B	C	minterm	F
0	0	0	m0 $\overline{A}\,\overline{B}\,\overline{C}$	0
0	0	1	m1 $\overline{A}\,\overline{B}C$	1
0	1	0	m2 $\overline{A}B\overline{C}$	1
0	1	1	m3 $\overline{A}BC$	1
1	0	0	m4 $A\overline{B}\,\overline{C}$	0
1	0	1	m5 $A\overline{B}C$	0
1	1	0	m6 $AB\overline{C}$	0
1	1	1	m7 ABC	0

Sum of minterms form (4)

- What is $A(\overline{B}+C)$ in sum of minterms form?

Maxterms

A	B	C	maxterm
0	0	0	M0 $A+B+C$
0	0	1	M1 $A+B+\overline{C}$
0	1	0	M2 $A+\overline{B}+C$
0	1	1	M3 $A+\overline{B}+\overline{C}$
1	0	0	M4 $\overline{A}+B+C$
1	0	1	M5 $\overline{A}+B+\overline{C}$
1	1	0	M6 $\overline{A}+\overline{B}+C$
1	1	1	M7 $\overline{A}+\overline{B}+\overline{C}$

- A sum term in which all variables appear once, either complemented or uncomplemented.
- Each maxterm evaluates to 0 for exactly one variable assignment, 1 for all others.
- Denoted by MX where X corresponds to the variable assignment for which MX = 0.

Relationship between minterms and maxterms

- Minterms and maxterms with the same subscripts are complements.

$$\overline{mX} = MX$$

Product of maxterms form

- The logical AND of all maxterms for which F = 0.

A	B	C	maxterm	F
0	0	0	M0 $A+B+C$	0
0	0	1	M1 $A+B+\overline{C}$	1
0	1	0	M2 $A+\overline{B}+C$	1
0	1	1	M3 $A+\overline{B}+\overline{C}$	1
1	0	0	M4 $\overline{A}+B+C$	0
1	0	1	M5 $\overline{A}+B+\overline{C}$	0
1	1	0	M6 $\overline{A}+\overline{B}+C$	0
1	1	1	M7 $\overline{A}+\overline{B}+\overline{C}$	0

$$F = (A+B+C)\,(\overline{A}+B+C)\,(\overline{A}+B+\overline{C})\,(\overline{A}+\overline{B}+C)\,(\overline{A}+\overline{B}+\overline{C})$$

$$= (M0)\,(M4)\,(M5)\,(M6)\,(M7)$$

$$= \prod M(0,4,5,6,7)$$

Product of maxterms form (2)

- The logical AND of all maxterms for which F = 0.

A	B	C	maxterm		F
0	0	0	M0	$A+B+C$	0
0	0	1	M1	$A+B+\overline{C}$	1
0	1	0	M2	$A+\overline{B}+C$	1
0	1	1	M3	$A+\overline{B}+\overline{C}$	1
1	0	0	M4	$\overline{A}+B+C$	0
1	0	1	M5	$\overline{A}+B+\overline{C}$	0
1	1	0	M6	$\overline{A}+\overline{B}+C$	0
1	1	1	M7	$\overline{A}+\overline{B}+\overline{C}$	0

M0	M1	M2	M3	M4	M5	M6	M7
0	1	1	1	1	1	1	1
1	0	1	1	1	1	1	1
1	1	0	1	1	1	1	1
1	1	1	0	1	1	1	1
1	1	1	1	0	1	1	1
1	1	1	1	1	0	1	1
1	1	1	1	1	1	0	1
1	1	1	1	1	1	1	0

Product of maxterms form (3)

- What is $\bar{F}$ in product of maxterms form?

A	B	C	maxterm		F
0	0	0	M0	$A+B+C$	0
0	0	1	M1	$A+B+\bar{C}$	1
0	1	0	M2	$A+\bar{B}+C$	1
0	1	1	M3	$A+\bar{B}+\bar{C}$	1
1	0	0	M4	$\bar{A}+B+C$	0
1	0	1	M5	$\bar{A}+B+\bar{C}$	0
1	1	0	M6	$\bar{A}+\bar{B}+C$	0
1	1	1	M7	$\bar{A}+\bar{B}+\bar{C}$	0

Summary of forms

	F	$\overline{F}$
Sum of minterms	$\sum m(F = 1)$	$\sum m(F = 0)$
Product of maxterms	$\prod M(F = 0)$	$\prod M(F = 1)$

POS & SOP

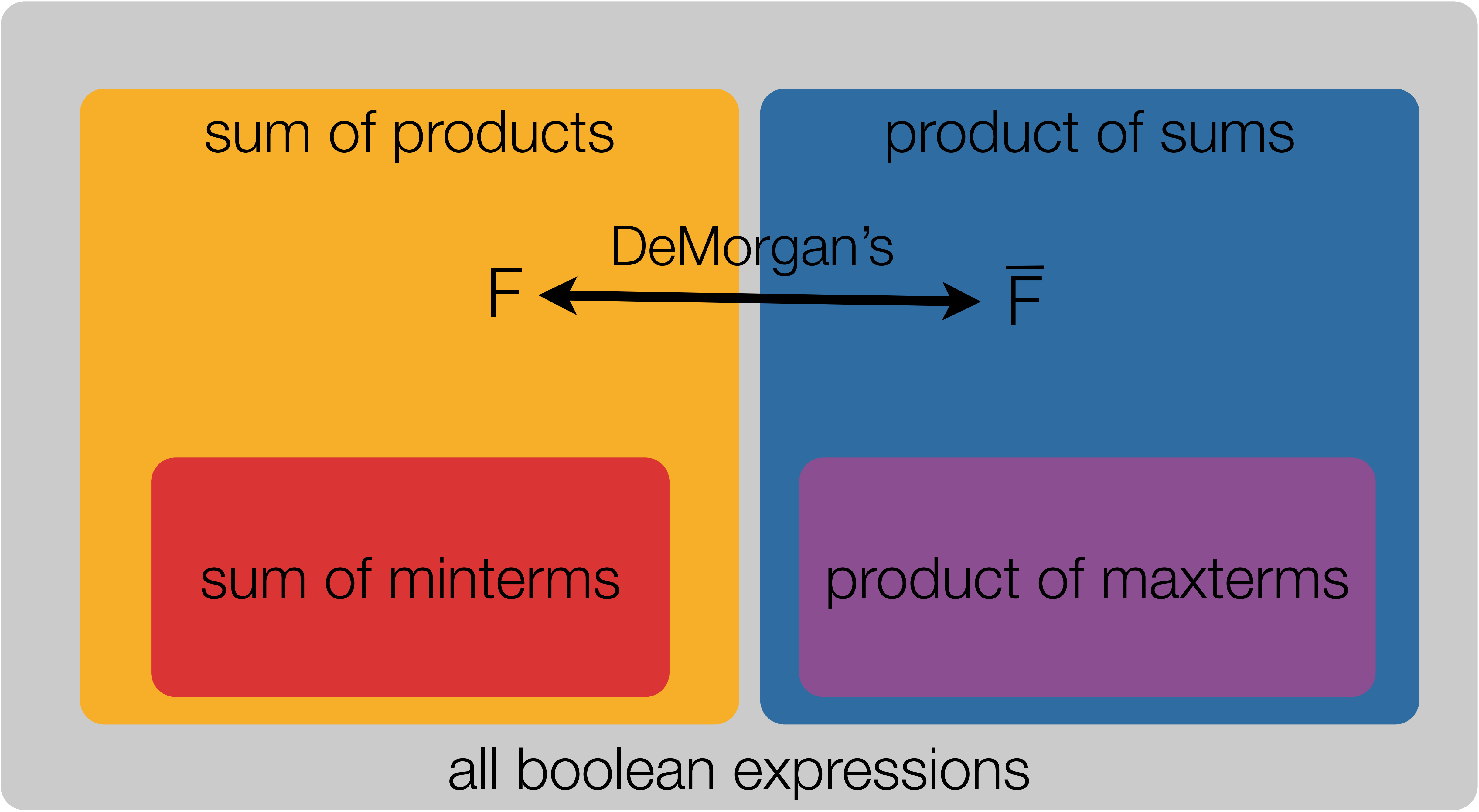
- Sum of products (SOP): OR of ANDs

$$\text{e.g., } F = \overline{Y} + \overline{X}Y\overline{Z} + XY$$

- Product of sums (POS): AND of ORs

$$\text{e.g., } G = X(\overline{Y} + Z)(X + Y + \overline{Z})$$

Relations between standard forms



Expression simplification / circuit optimization

Cost criteria

- Literal cost: the number of literals in an expression
- Gate-input cost: the literal cost + all terms with more than one literal + (optionally) the number of distinct, complemented single literals

Roughly proportional to the number of transistors and wires in an AND/OR/NOT circuits. Does not apply, to more complex gates, for example XOR.

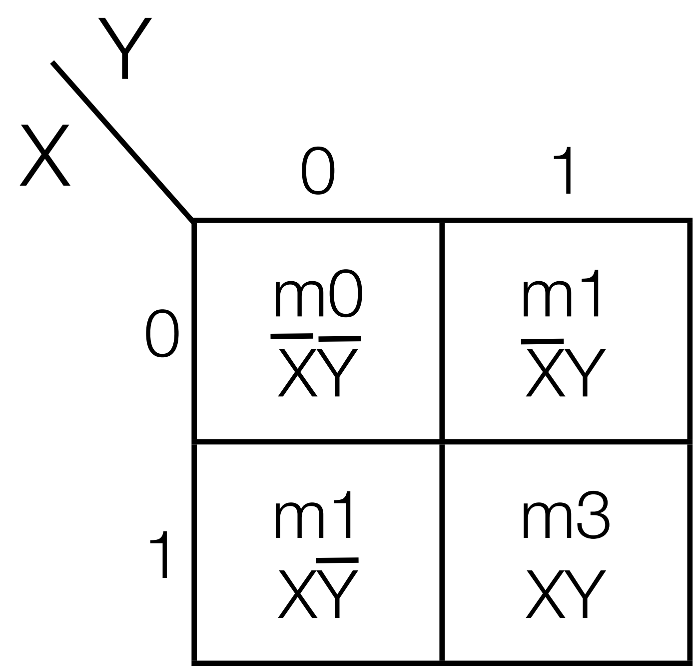
	Literal cost	Gate-input cost
$G = \bar{A}\bar{B}\bar{C}\bar{D} + ABCD$	8	8 + 2 + (4)
$G = (\bar{A}+B)(\bar{B}+C)(\bar{C}+D)(\bar{D}+A)$	8	8 + 4 + (4)

Karnaugh maps

- All functions can be expressed with a map

- There is one square in the map for each minterm in a function's truth table

X	Y	F
0	0	m0
0	1	m1
1	0	m2
1	1	m3



Karnaugh maps express functions

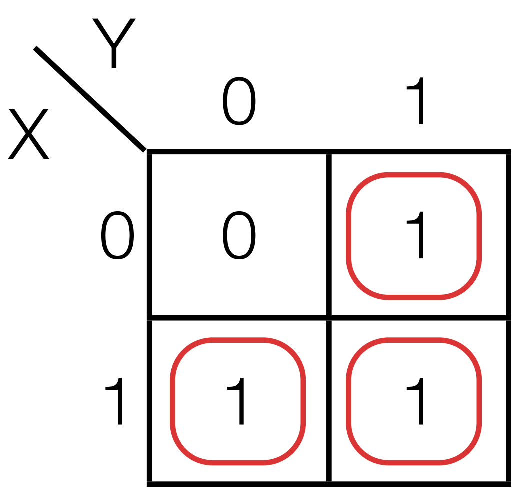
- Fill out table with value of a function

X	Y	F
0	0	0
0	1	1
1	0	1
1	1	1

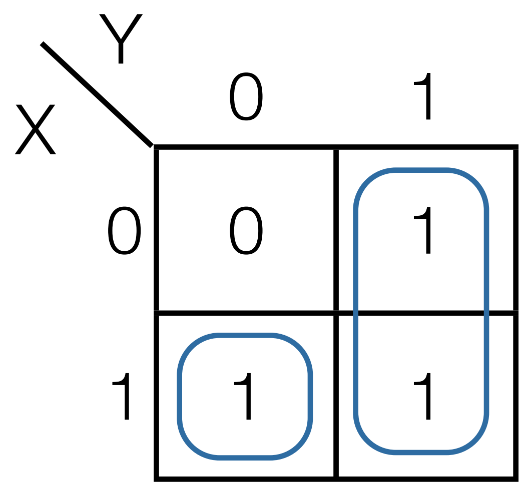
X \ Y	0	1
0	0	1
1	1	1

Simplification using a k-map

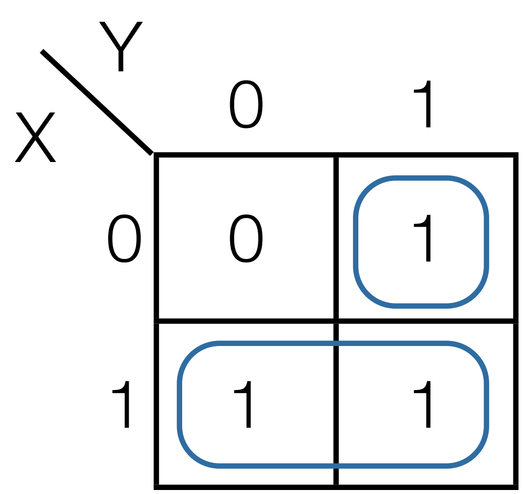
- Whenever two squares share an edge and both are 1, those two terms can be combined to form a single term with one less variable



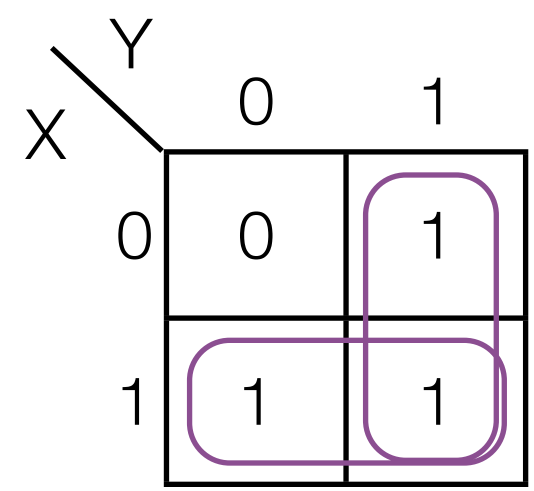
$F = \overline{X}Y + X\overline{Y} + XY$



$F = Y + X\overline{Y}$



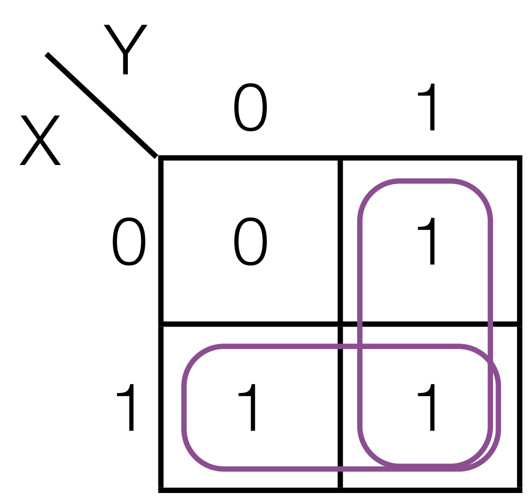
$F = X + \overline{X}Y$



$F = X + Y$

Simplification using a k-map (2)

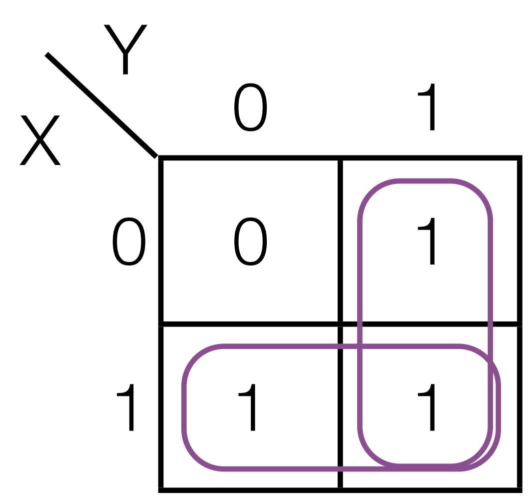
- Circle contiguous groups of 1s (circle sizes must be a power of 2)
- There is a correspondence between circles on a k-map and terms in a function expression
- The bigger the circle, the simpler the term
- Add circles (and terms) until all 1s on the k-map are circled



$$F = X + Y$$

Karnaugh maps: terminology

- A term is an **implicant** if it has the value 1 for all minterms (corresponds to any circled groups of 1, 2, 4, etc. 1s on a k-map)
- A term is a **prime implicant** if the removal of any literal makes it no longer an implicant (corresponds to circles that cannot be made any larger)
- If a minterm is included in only one prime implicant, that implicant is called an **essential prime implicant** (corresponds to any circle that is the only one to cover a 1 on a k-map)



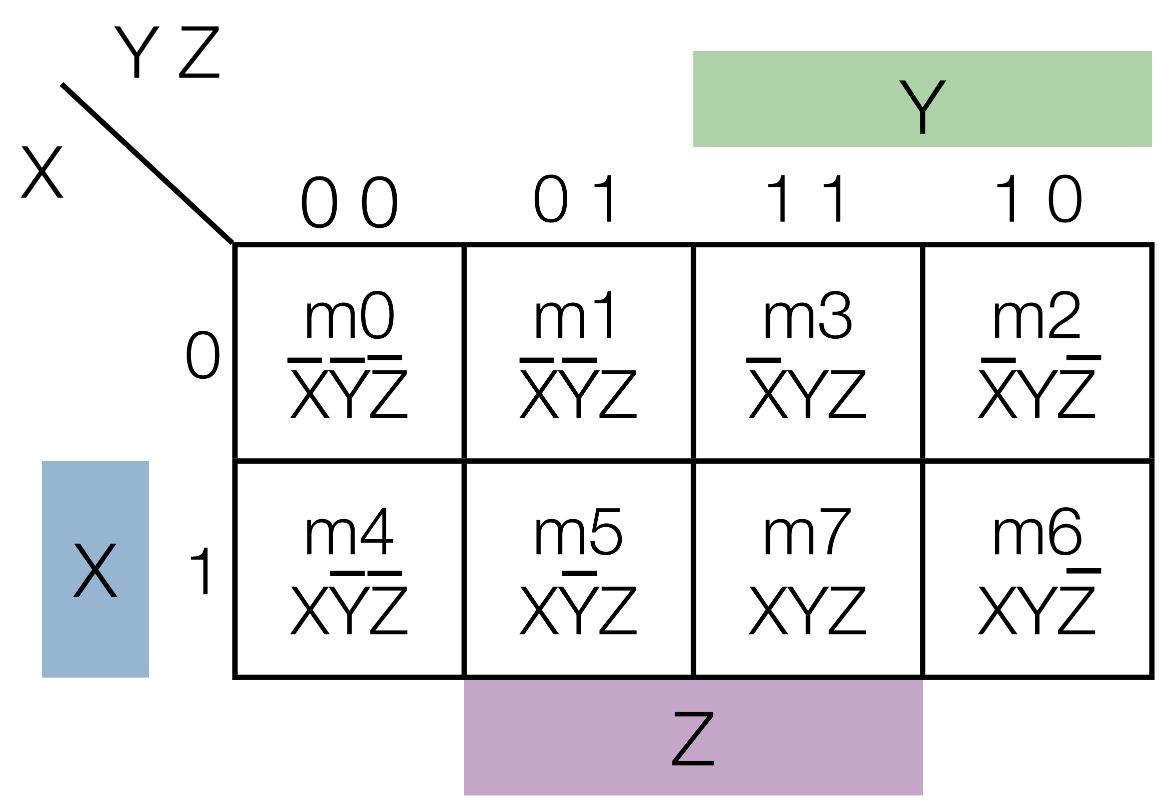
$F = X + Y$

Karnaugh-map example

X	Y	F
0	0	1
0	1	1
1	0	0
1	1	0

3-variable Karnaugh maps

- Use gray ordering on edges with multiple variables
- Gray encoding: order of values such that only one bit changes at a time
- Two minterms are considered adjacent if they differ in only one variable (this means maps wrap)



3-variable Karnaugh maps (2)

- List all all of the prime implicants for this function
- Is any of them an essential prime implicant?
- What is a simplified expression for this function?

X \ YZ	00	01	11	10
0	1	1	0	1
1	1	1	1	1

(Y spans columns 11 and 10; X spans row 1; Z spans columns 01 and 11)

4-variable Karnaugh maps

- Extension of 3-variable maps

WX \ YZ	0 0	0 1	1 1	1 0
0 0	m0	m1	m3	m2
0 1	m4	m5	m7	m6
1 1	m12	m13	m15	m14
1 0	m8	m9	m11	m10

Y (columns 1 1, 1 0) · X (rows 0 1, 1 1) · W (rows 1 1, 1 0) · Z (columns 0 1, 1 1)

4-variable Karnaugh maps (2)

- List all all of the prime implicants for this function
- Is any of them an essential prime implicant?
- What is a simplified expression for this function?

WX \ YZ	0 0	0 1	1 1	1 0
0 0	1	1	0	1
0 1	1	1	0	1
1 1	1	1	0	0
1 0	1	1	0	1

Y (columns 11, 10); X (rows 01, 11); W (rows 11, 10); Z (columns 01, 11)

4-variable Karnaugh maps (3)

- List all all of the prime implicants for this function
- Is any of them an essential prime implicant?
- What is a simplified expression for this function?

WX \ YZ	0 0	0 1	1 1	1 0
0 0	0	0	1	0
0 1	1	1	1	0
1 1	0	1	1	1
1 0	0	1	0	0

5-variable Karnaugh maps

A = 0 (labels: D = columns 11, 10; E = columns 01, 11; C = rows 01, 11; B = rows 11, 10)

BC \ DE	0 0	0 1	1 1	1 0
0 0	m0	m1	m3	m2
0 1	m4	m5	m7	m6
1 1	m12	m13	m15	m14
1 0	m8	m9	m11	m10

A = 1 (labels: D = columns 11, 10; E = columns 01, 11; C = rows 01, 11; B = rows 11, 10)

BC \ DE	0 0	0 1	1 1	1 0
0 0	m16	m17	m19	m18
0 1	m20	m21	m23	m22
1 1	m28	m29	m31	m30
1 0	m24	m25	m27	m26

5-variable Karnaugh map (2)

- What is a simplified expression for this function?

A = 0

BC \ DE	00	01	11	10
00	0	0	1	1
01	1	1	1	1
11	1	1	0	0
10	0	1	1	0

(D: columns 11, 10; E: columns 01, 11; C: rows 01, 11; B: rows 11, 10)

A = 1

BC \ DE	00	01	11	10
00	0	0	1	1
01	1	1	0	0
11	1	1	0	0
10	0	1	1	0

(D: columns 11, 10; E: columns 01, 11; C: rows 01, 11; B: rows 11, 10)

Design example : 2-bit multiplier

a1	a0	b1	b0	z3	z2	z1	z0
0	0	0	0				
0	0	0	1				
0	0	1	0				
0	0	1	1				
0	1	0	0				
0	1	0	1				
0	1	1	0				
0	1	1	1				
1	0	0	0				
1	0	0	1				
1	0	1	0				
1	0	1	1				
1	1	0	0				
1	1	0	1				
1	1	1	0				
1	1	1	1				

K-Maps: Complements, POS, don’t care conditions

Finding $\overline{F}$

- Find prime implicants corresponding to the 0s on a k-map

WX \ YZ	00	01	11	10
00	1	1	0	1
01	1	1	0	1
11	1	1	0	0
10	1	1	0	1

$$F = \overline{Y} + \overline{X}\,\overline{Z} + \overline{W}\,\overline{Z}$$

WX \ YZ	00	01	11	10
00	0	0	1	0
01	0	0	1	0
11	0	0	1	1
10	0	0	1	0

$$\overline{F} = YZ + WXY$$

POS expressions from a k-map

- Find $\overline{F}$ as SOP and then apply DeMorgan's

WX \ YZ	00	01	11	10
00	1	1	0	1
01	1	0	0	0
11	1	0	0	0
10	1	1	0	1

$$\overline{F} = YZ + XZ + YX$$

DeMorgan's

$$F = (\overline{Y}+\overline{Z})(\overline{Z}+\overline{X})(\overline{Y}+\overline{X})$$

Optimized standard forms Example

- M&K 2-26 (b)

- Find optimized versions of F as SOP and as POS:

$$F(W,X,Y,Z) = \sum m(3,4,9,15)$$

$$d(W,X,Y,Z) = \sum m(0,1,2,5,10,14)$$

Don't care conditions

- There are circumstances in which the value of an output doesn't matter
- For example, in that 2-bit multiplier, what if there were only 3 bits for the product and one bit to indicate an overflow situation?
- Don't care situations are denoted by "x" in a truth table and in Karnaugh maps.
- Can also be expressed in minterm form:

$$z2 = \sum m(10,11,14)$$
$$d = \sum m(15)$$

- During minimization can be treated as either a 1 or a 0

a1	a0	b1	b0	o	z2	z1	z0
0	0	0	0	0	0	0	0
0	0	0	1	0	0	0	0
0	0	1	0	0	0	0	0
0	0	1	1	0	0	0	0
0	1	0	0	0	0	0	0
0	1	0	1	0	0	0	1
0	1	1	0	0	0	1	0
0	1	1	1	0	0	1	1
1	0	0	0	0	0	0	0
1	0	0	1	0	0	1	0
1	0	1	0	0	1	0	0
1	0	1	1	0	1	1	0
1	1	0	0	0	0	0	0
1	1	0	1	0	0	1	1
1	1	1	0	0	1	1	0
1	1	1	1	1	x	x	x

Don't care example

- M&K 2-24 (a)

- Optimize this function:

$$F(A,B,C,D) = \sum m(0,1,7,13,15)$$

$$d(A,B,C,D) = \sum m(2,6,8,9,10)$$

Glitches and hazards

- Glitch: an unintended change in circuit output
- Hazard: the hardware structures that cause a glitch to occur
- Caused by multiple path delays through a circuit
- Example: $\bar{A}\bar{B} + BC$
- Avoidance
  - Synchronous design (coming later)
  - Extra implicants

Next week: multibit outputs and standard circuits