Appendix: Mathematical Models and Insight

Group 6

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1 Optimal speeds for best finishing time in single player mode

The R players start our aligned one behind the other and move in a column, at speed v_1, until the first player gets exhausted. Let this occur at time t_1, after having covered distance d_1. Then they continue with speed v_2 until the second rider (the column leader) tires out. This happened after an additional time t_2 and distance d_2. Continue like this, dropping a rider each time, until the last rider exhausts his energy.

For the first interval, the column leader will expend an enery equal to $v_1^{2.5} t_1$, which we know is E, since the rider started ou with E and now has no energy left. So $t_1 = \frac{E}{v_1^{2.5}}$, and $d_1 = v_1 \cdot t_1 = \frac{E}{v_1^{1.5}}$. At time t_1, since all the riders behind the column leader had used $\frac{7}{10} v^{2.5}$ energy per unit time, they will have expanded $\frac{7}{10}E$. Everybody will then have energy $\frac{3}{10}E$ left as they ride into the second stretch. Replacing E with $\frac{3}{10}E$, we get that the second rider to drop out will do so after another $t_2 = \frac{3}{10}\frac{E}{v_2^{2.5}}$ and $d_2 = \frac{3}{10}\frac{E}{v_2^{1.5}}$. In general, the time difference between when the $k-1^{st}$ and k^{th} rider drop out is

$$t_k = (\frac{3}{10})^{k-1} \frac{E}{v_k^{2.5}} \tag{1}$$

and the extra distance travelled is

$$d_k = (\frac{3}{10})^{k-1} \frac{E}{v_k^{1.5}} \tag{2}$$

In particular, the last rider, rider R, will ride $t_R = (\frac{3}{10})^{R-1} \frac{E}{v_R^{2.5}}$ and $d_R = (\frac{3}{10})^{R-1} \frac{E}{v_R^{1.5}}$ by himself, before tiring out. Summing up (2) over all intervals, the total distance covered by him will be:

$$\bar{D} = \sum_{k=1}^{R} d_k = \sum_{k=1}^{R} (\frac{3}{10})^{k-1} \frac{E}{v_k^{1.5}} \tag{3}$$

Similarly, his total time will be:

$$\bar{T} = \sum_{k=1}^{R} d_k = \sum_{k=1}^{R} (\frac{3}{10})^{k-1} \frac{E}{v_k^{2.5}} \tag{4}$$

We want to minimize $\bar{T}$. But we also want our last rider to reach the finishing line. Thus, $\bar{D} \geq D$. Since finishing with extra energy serves no purpose, we can restrict this to $\bar{D} = D$, without fear of losing optimal solutions. This will constitute a constraint on the optimization problem. The other constraint is $0 \leq v_k \leq 25$ $\forall k$. Thus, **we want to pick v_k, $k = 1, \ldots, R$ which minimize $\bar{T}$ subject to $\bar{D} = D$.**

We can express v_1 from (3) and plug it into (4):

$$v_1 = [\frac{D}{E} - \sum_{k=2}^{R} (\frac{3}{10})^{k-1} \frac{1}{v_k^{1.5}}]^{-\frac{2}{3}} \tag{5}$$

So

$$\bar{T} = \frac{E}{v_1^{2.5}} + \sum_{k=2}^{R} (\frac{3}{10})^{k-1} \frac{E}{v_k^{2.5}} = E[\frac{D}{E} - \sum_{k=2}^{R} (\frac{3}{10})^{k-1} \frac{1}{v_k^{1.5}}]^{\frac{5}{3}} + \sum_{k=2}^{R} (\frac{3}{10})^{k-1} \frac{E}{v_k^{2.5}}$$

We can thus view $\bar{T}$ as a function of $v_2, v_3, \ldots, v_R$ over $[0..25]^{R-1}$ and look for minima for this function with no further constraints.

A minimum can occur either on the boundary, or inside the region $[0..25]^{R-1}$.

- Inside the region. We are searching for points where $\frac{\partial \bar{T}}{\partial v_k} = 0$ $\forall k = 2, \ldots, R$. We can express this as:

$$\frac{\partial \bar{T}}{\partial v_k} = \frac{5}{3} E[\frac{D}{E} - \sum_{k=2}^{R} (\frac{3}{10})^{k-1} \frac{1}{v_k^{1.5}}]^{\frac{2}{3}} (\frac{3}{10})^{k-1} \frac{3}{2} \frac{1}{v_k^{2.5}} - \frac{5}{2} (\frac{3}{10})^{k-1} \frac{E}{v_k^{3.5}} = 0$$

  rearranging and simplifying,

$$[\frac{D}{E} - \sum_{k=2}^{R} (\frac{3}{10})^{k-1} \frac{1}{v_k^{1.5}}]^{\frac{2}{3}} = \frac{1}{v_k}$$

  This is an amazing fact: All speeds v_k will equal to some common $\bar{v}$ at an optimum solution, since the LHS is the same for all k! We include $k = 1$ here as well, because of (5). (Thinkig a bit, this fact should not be so surprising, given the fact that after the first rider drops out, the situation on the field is the same as at the beginning, except everybody starts out with a 3/10 fraction of the original energy.)

  To find this $\bar{v}$, we need to solve

$$[\frac{D}{E} - \sum_{k=2}^{R} (\frac{3}{10})^{k-1} \frac{1}{\bar{v}^{1.5}}]^{\frac{2}{3}} = \frac{1}{\bar{v}}$$

  or

$$[\frac{D}{E} - \sum_{k=2}^{R} (\frac{3}{10})^{k-1} \frac{1}{\bar{v}^{1.5}}] = \frac{1}{\bar{v}^{1.5}}$$

  Denote $x = \frac{1}{\bar{v}^{1.5}}$. We thus need to solve

$$\frac{D}{E} - x \sum_{k=2}^{R} (\frac{3}{10})^{k-1} = x$$

or

$$\frac{D}{E} - x(\frac{1-(3/10)^R}{1-3/10} - 1) = x$$

Hence,

$$x\frac{10}{7}(1-(\frac{3}{10})^R) = \frac{D}{E}$$

and using the definition of x,

$$\frac{1}{\bar{v}^{1.5}}\frac{10}{7}(1-(\frac{3}{10})^R) = \frac{D}{E}$$

We thus arrive at

$$\boxed{\bar{v} = [\frac{10}{7}\frac{E}{D}(1-(\frac{3}{10})^R)]^{\frac{2}{3}} \qquad (6)}$$

- On the boundary. Then at least one of the values v_k is either 0 or 25. Rather than extending the analysis for such cases, we use some practical intuition. The function $\bar{T}$ of variables $(v_2, \ldots, v_R)$ is convex, and we expect that, if the minimum point found above is inside the region $[0..25]^{R-1}$, then that will the global minimum of the function. On the other hand, if the point lies outside (i.e., $\bar{v} > 25$), then it means that 25 is a sustainable speed, and the column can afford to go at the maximum speed. This is clearly optimal in terms of finishing time. If it wasn't obvious, we note that v_k is at no point negative: rational (non-integral) powers of negative numbers are undefined. Mathematics aside, it makes no sense for the column to go backwards at any point in the race, so we don't lose any useful solutions by restricting $v > 0$.

2 Optimal starting strategy

The above analysis assumes we can form a column instantaneously. That is far from reality, the starting strategy can be crucial to the success of the team. In hindsight, the fact that the tournament was run on a very short $L = 1800$ board validates our concern to devote significant attention to the starting phase of the game.

There was a strong debate both in class and amongst ourselves as to which the best approach is:

- stay at the starting line and form the column there, and start advancing only after the column is fully formed;
- let all riders start biking as fast as they can towards reaching the optimal speed, and converge into the column as they go;
- some strategy in between the two extremes outlined above.

To gain some insight into the problem, we used a simple mathematical model suitable for accurate analysis, and that would hopefully guide us to choose among possible initial strategies.

Specifically, we consider only two riders, placed a distance p apart at the starting line. Furthermore, we assume that the riders can switch instantly to any speed they want. The riders will try to converge as fast as possible, so they will move at each step one lane towards one another. Henve, it will take them $\tilde{T} = \frac{p}{2}$ time to meet (the lane-switching speed is 1). Let the meeting point be d away from the starting line (figure 1).

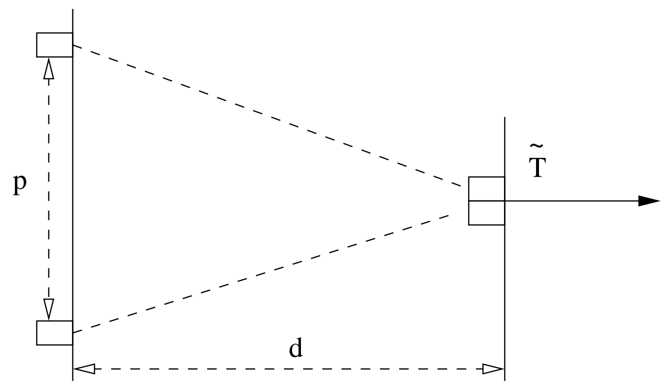

Figure 1: Riders start p apart and converge at time $\tilde{T}$

As seen in the figure, we neglect second order effects such as the parity of p and how they will arrange themselves into the column. Since the problem is symmetric in the two riders, we assume they use the same velocities. The energy at the time the riders meet is

$$\tilde{E} = E - \int_0^{\tilde{T}} v(t)^{2.5} dt$$

Once they meet, they will of course go with the optimal speed computed at the meeting point, according to (6):

$$\bar{v} = [\frac{\tilde{E}}{D-d} \frac{10}{7} (1 - (\frac{3}{10})^2)]^{\frac{2}{3}}$$

We wish to choose the speed function v up to the meeting point to minimize the total time of the race, which is the time taken until convergence, plus the time taken from that point until the end of the

$$T = \tilde{T} + \frac{D-d}{\bar{v}}$$

Recalling $\tilde{T} = \frac{p}{2}$, and using $d = \int_0^{\tilde{T}} v(t)dt$, our target T to minimize can be expressed as below. To keep the formula complexity low, we combine all the constant terms (not containing v) into a generic constants K_1, K_2:

$$T = \frac{p}{2} + \frac{D-d}{[\frac{\tilde{E}}{D-d} \frac{10}{7} (1 - (\frac{3}{10})^2)]^{\frac{2}{3}}} = K_1 + K_2 \frac{(D-d)^{\frac{5}{3}}}{\tilde{E}^{\frac{2}{3}}} = K_1 + K_2 \frac{(D - \int_0^{\frac{p}{2}} v(t)dt)^{\frac{5}{3}}}{(E - \int_0^{\frac{p}{2}} v(t)^{2.5} dt)^{\frac{2}{3}}}$$

We want to minimize T. To put the formula into perspective, we make the observation that if $v = 0$, then they will meet on the starting line, which is the first strategy discussed. The riders will waste more time, but save energy by not riding in full wind. This energy will translate into higer speed for the rest of the race. As v increases, the strategy turns into the second strategy, where the riders save time by riding fast, but waste more energy since they ride alone with high speeds until they meet into the column. The math will tell us which is a better approach if all we care is the total time of the race.

To make the above easier to analyze, we make the simplifying assumption that v is constant up to time $\tilde{T} = \frac{p}{2}$. Then

$$T = K_1 + K_2 \frac{(D - \frac{p}{2}v)^{\frac{5}{3}}}{(E - \frac{p}{2}v^{2.5})^{\frac{2}{3}}}$$

We want to minimize the function

$$f(v) = \frac{(D - \frac{p}{2}v)^{\frac{5}{3}}}{(E - \frac{p}{2}v^{2.5})^{\frac{2}{3}}}$$

After some basic calculus that we spare the reader, v' optimal turns out to be $(\frac{E}{D})^{\frac{2}{3}}$. What is this v'? It is precisely the optimal speed (formula (6)) for the case $R = 1$.

Thus, to minimize the total time, the two riders should ride with v', the optimal speed for $R = 1$ until they meet, and then switch to $\bar{v}$, the optimal speed for $R = 2$. Our intuition is satisfied. Note that there were some simplifying assumptions, mainly that the riders can start at v' from the very beginning. It is not too much of a stretch to ignore this, and assume that if it is optimal to ride with v' until meeting, then starting at 0, accelerating up to v' and then riding at v' until meeting, is also optimal, or close to optimal, for the real race conditions.

Hence, since v' is much closer to $\bar{v}$ than to 0, it seems like the second strategy was closer to optimal than the first strategy we mentioned at the beginning. In our player, we use the results from this simplified model, and generalize them to $R > 2$ riders.

3 Who should be at front?

This was a question that also came up, and intuitively, it seemed that if the column contains riders of different energies, then the rider of least energy ought to go at the front, and die out fast, and the rider of most energy (the one who will finish the race) should be spared the wind. There is no advantage in sparing a rider who will die out first at the expense of a rider who will outlive him. We validate this using the following formula whose derivation we leave as an exercise for the interested reader:

$$T = \frac{D^{\frac{5}{3}}}{(E_2 + 0.3E_1)^{\frac{2}{3}}}$$

This formula describes the total race time, if we start out with two riders, one of energy E_1, and another of energy E_2, and rider 1 is in front of rider 2. Swapping the place of the riders will decrease the finishing time if and only if $E_1 + 0.3E_2 > E_2 + 0.3E_1 \iff 0.7E_1 > 0.7E_2 \iff E_1 > E_2$. So the rider of lowest energy should always be placed at the front of the column.

4 Marginal benefit from having an extra rider

Formula (6) states that the more riders we have (the larger R is), the higher the optimal speed $\bar{v}$ is. This value was derived under a simplified model, and costs such as that associated with forming the column have not been taken into account. As we get more and more riders, the marginal benefit to the optimal speed get smaller and smaller, whereas the overhead of forming the column increases roughly linearly with the number of riders.

Let us take then a closer look at the effect that R has on the race time. Recall, roughly $T = \frac{D}{\bar{v}}$. Viewing everything as a function of R, we write

$$dT = -\frac{Dd\bar{v}}{\bar{v}^2} = -\frac{D\frac{2}{3}[\frac{10}{7}\frac{E}{D}(1-(\frac{3}{10})^R)]^{\frac{-1}{3}}\frac{10}{7}\frac{E}{D}\ln(\frac{3}{10})(\frac{3}{10})^R}{\bar{v}^2} = \frac{\frac{20}{21}E[\frac{10}{7}\frac{E}{D}(1-(\frac{3}{10})^R)]^{\frac{-1}{3}}\ln(\frac{10}{3})(\frac{3}{10})^R}{\bar{v}^2}$$

Since $\bar{v} = [\frac{10}{7}\frac{E}{D}(1-(\frac{3}{10})^R)]^{\frac{2}{3}}$, we can rewrite

$$dT = \frac{\frac{20}{21}E\bar{v}^{-0.5}\ln(\frac{10}{3})(\frac{3}{10})^R}{\bar{v}^2} = \frac{\frac{20}{21}E\ln(\frac{10}{3})(\frac{3}{10})^R}{\bar{v}^{2.5}}$$

We can approximate $\bar{v} = (\frac{E}{D})^{\frac{2}{3}}$ to get

$$dT \simeq \frac{D}{v}\frac{20}{21}\ln(\frac{10}{3})(\frac{3}{10})^R \simeq 1.15(\frac{3}{10})^R \cdot T$$

Thus, we get an improvement in finishing time that is exponentially decreasing with R. To gain a feel for the numbers involed, consider the race parameters that we used almost exclusively in class: $R = 4$, $T \simeq 15,000$ (as it turned out; also verifiable by computation). $1.15(\frac{3}{10})^R \simeq 0.009 < 1\%$. Hence, adding a fifth player improves finishing time by 1%, regardless of any of the other parameters of the race. For the T above, this is 150 time units. In comparison, if $R = 8$, $dT \simeq 75 \times 10^{-6}$. For the T in the race we ran, this amounts to 1.13 time units. To put the result into perspective, adding a nith rider to teams improves the total time by a single step of the simulator! The overhead to put 9 riders into a column most certainly exceeds this benefit. Hence, teams are better of using a column of fewer players and doing something else with the extra riders, such as blocking other teams.

This effect was observed, to a lesser degree, when running the race with $R = 6$ in class, and no improvement in finishing time was noticed.