AN EULER-GENUS APPROACH TO THE CALCULATION OF THE CROSSCAP-NUMBER POLYNOMIAL

YICHAO CHEN AND JONATHAN L. GROSS

Abstract. In 1994, J. Chen, J. Gross, and R. Rieper demonstrated how to use the rank of Mohar's overlap matrix to calculate the crosscap-number distribution, that is, the distribution of the embeddings of a graph in the non-orientable surfaces. That has ever since been by far the most frequent way that these distributions have been calculated. This paper introduces a way to calculate the *Euler-genus polynomial* of a graph, which combines the orientable and the non-orientable embeddings, without using the overlap matrix. The crosscap-number polynomial for the non-orientable embeddings is then easily calculated from the Euler-genus polynomial and the genus polynomial.

A surface with Euler characteristic c is said to have ***Euler genus*** $2-c$. Thus, the Euler genus is the crosscap number of any non-orientable surface and twice the genus of any orientable surface. For the most part, the Euler genus has previously been something to be calculated from the genus and the crosscap number.

Heretofore, the usual way to calculate the crosscap-number distribution for a graph has been, as initiated by [3], by means of the overlap matrix [33]. In this paper, we provide a new three-step way to calculate crosscap-number distribution. The ***Euler-genus polynomial of a graph*** is the generating function for the numbers of embeddings of that graph, according to the Euler genera of surfaces. If the genus polynomial is known, then from it and the Euler-genus polynomial, we readily obtain the crosscap-number polynomial, as per Theorem 2.1. A key ingredient to our three-step method is directly calculating the Euler-genus polynomial, by solving a recursion

Calculating embedding distributions is an enumerative branch of topological graph theory. For the connections of calculating embedding distributions with physics and other areas of mathematics, we refer the reader to [29, 30, 34] etc.

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1. Basic Terminology and Background

In this section, we provide some of the basic concepts and notations of topological graph theory. We also cite some of the prior results relevant to this paper.

A ***graph*** $G = (V(G), E(G))$ is permitted to have loops and multiple edges. We use S to denote a surface without regard to orientability. In this paper, all graph embeddings are *cellular embeddings*.

A ***rotation at a vertex*** v of a graph G is a cyclic ordering of the edge-ends incident at v. A ***(pure) rotation system*** ρ of a graph G is an assignment of a rotation at every vertex of G. It is well-known that there is an one-to-one correspondence between the rotation systems and orientable embeddings.

A ***general rotation system*** for a graph G is a pair (ρ, λ), where ρ is a rotation system and λ is a map on $E(G)$ with values in $\{0, 1\}$. If $\lambda(e) = 1$, then the edge e is said to be ***twisted***; otherwise $\lambda(e) = 0$, and we call the edge e ***untwisted***. It is obvious that if $\lambda(e) = 0$, for all $e \in E(G)$, then the general rotation system (ρ, λ) is equivalent to a pure rotation system. Recall that any embedding of G into a surface S can be described by a general rotation system [38]. We call λ a ***twist-indicator***.

For further background in topological graph theory consistent with this paper, we recommend [22] and [46].

1.1. Polynomials that enumerate embeddings.

As usual in topological graph theory, we denote the closed orientable surface of genus g by S_g and the closed non-orientable surface of crosscap number k by N_k. We recall that the Euler characteristic of the surface S_g is $2-2g$ and that the Euler characteristic of the surface N_k is $2 - k$.

We observe that the Klein bottle N_2 and the torus S_1 both have 0 as their Euler characteristic. In general, we observe that the operation of adding a single handle to a surface has the same effect on the Euler characteristic as the operation of adding two crosscaps, namely, subtracting 2. Accordingly, these two operations have the same effect on Euler-genus. (Indeed, if the initial surface is non-orientable, then these two operations have the same effect on isomorphism type.)

For $g \geq 0$, we use $\gamma_G(g)$ to denote the number of embeddings of G in the surface S_g. The ***genus distribution*** of a graph G is the sequence

$$\gamma_G(0),\ \gamma_G(1),\ \gamma_G(2),\ \ldots\ .$$

The ***genus polynomial*** of G is defined as the generating function for the genus distribution sequence $\gamma_G(n)$:

$$\Gamma_G(x) \,=\, \sum_{i=0}^{\infty} \gamma_G(i)x^i.$$

For $k \geq 1$, we use $\tilde{\gamma}_G(k)$ to denote the number of embeddings of G in the surface N_k. Analogous to genus distribution, the ***crosscap-number distribution*** of a graph G is the sequence

$$\widetilde{\gamma}_G(1),\ \widetilde{\gamma}_G(2),\ \widetilde{\gamma}_G(3),\ \ldots,$$

and the ***crosscap-number polynomial*** of G is defined as

$$\mathcal{X}_G(y) \,=\, \sum_{j=1}^{\infty} \widetilde{\gamma}_G(j)y^j.$$

For any graph $G = (V(G), E(G))$, the number of embeddings of G in a surface of Euler-genus i is denoted by $\varepsilon_G(i)$. If i is odd, then $\varepsilon_G(i)$ equals the number of embeddings of G in the surface N_i. If i is even, then $\varepsilon_G(i)$ equals the sum of the number of embeddings of G in N_i plus the number of embeddings in $S_{i/2}$.

The ***Euler-genus distribution*** of the graph G is the sequence

$$\varepsilon_G(0),\ \varepsilon_G(1),\ \varepsilon_G(2),\ \ldots\ .$$

The ***Euler-genus polynomial*** is the generating function

$$\mathcal{E}_G(y) \,=\, \sum_{i=0}^{\infty} \varepsilon_G(i)y^i\ .$$

We let $F(\iota)$ denote the number of faces of an embedding $\iota : G \to S$, and we let $\gamma^E(S)$ denote the Euler-genus of S. Then, in terms of Euler-genus, the classical Euler polyhedral equation is

$$|V(G)| - |E(G)| + |F(\iota)| \,=\, 2 - \gamma^E(S).$$

1.2. **Some relevant background.** After Gross and Furst [15] introduced the genus distribution of a graph, many authors have calculated the genus distributions for various classes of graphs. For example, Gross et al. [12, 21] computed them for bouquets of circles, closed-end ladders and cobblestone path; Rieper [37], and Kwak and Lee [27] computed them independently for dipoles; McGeoch [31] for circular ladders and Möbius ladders; Tesar [43] for Ringel ladders; Stahl [40, 41] for H-linear families and some small diameter graphs; Wan and Liu [45] for some ladder-type graphs; and Mohar [34] for doubly hexagonal chains. For the recent advances on this topic, we refer the reader to [17, 19, 20, 34, 7, 5].

The theoretical foundations of crosscap-number distribution were established in [3] by J. Chen, Gross, and Rieper. Compared with the genus distribution, it is known for relatively few classes of graphs, see [3, 4, 11, 9, 10, 6] for more details. It seems that the calculation of crosscap-number distribution is more difficult than the calculation of genus distribution; the techniques used to derive crosscap-number distributions of graphs seem comparatively cumbersome [3], as illustrated by such a calculation for the crosscap-number distributions of necklaces, which can be found in Chapter 6 of J. Chen's thesis [2].

J. Chen, Gross and Rieper [3] subsequently discovered that rearranging the set of embeddings according to the rank of Mohar's algebraic invariant [33], called the ***overlap matrix***, sometimes facilitates the calculation of the total embedding distribution of a graph, and then calculated the total embedding distributions for necklaces, closed-end ladders, and cobblestone paths.

For a fixed spanning tree T, a T***-rotation system*** (ρ, λ) of G is a general rotation system (ρ, λ) such that $\lambda(e) = 0$, for every edge $e \in E(T)$. It is known [22] that there is a sequence of vertex-flips that transforms a general rotation system into a T-rotation system. Two embeddings of G are considered to be *equivalent* if their T-rotation systems are combinatorially equivalent.

Using the overlap matrix and ***Chebyshev polynomials of the second kind***, Chen and verious co-authors later extended the calculation of crosscap number distribution to other types of graphs, for example, to Ringel ladders [11] and circular ladders [6]. In such calculations, a key step is to find the correspondence between some T-rotation system and its overlap matrix. For example, the present authors and Mansour [6] used the *Gustin representation* of a 3-regular graph G to obtain a correspondence between a fixed overlap matrix of G and the T-rotation systems of G. When the degree of a vertex of G is larger than 3, it is not easy to find such a correspondence.

2. A Three-Step Procedure for the Crosscap Polynomial

We now introduce the ***Three-Step Procedure*** to calculate the crosscap-number distribution of a graph:

Step 1: Calculate the genus polynomial $\Gamma_G(x)$ for the graph G.
Step 2: Calculate the Euler-genus polynomial $\mathcal{E}_G(y)$ for the graph G.
Step 3: Calculate $\mathcal{X}_G(y) = \mathcal{E}_G(y) - \Gamma_G(y^2)$, using Equation (2.1).

Methods for achieving Step 1 have been developed over the past 30 years. We presently offer no new ideas for that step. Step 3 involves the application of the following simple observation:

Theorem 2.1. *The crosscap-number polynomial of a graph G satisfies the equality*

$$\mathcal{X}_G(y) = \mathcal{E}_G(y) - \Gamma_G(y^2). \quad \square \tag{2.1}$$

Remark The 3-step procedure given here explicitly was introduced implicitly in [7, 5] by the present authors, with Mansour and Tucker, in order to calculate genus polynomials for some linear families and some ring-like families.

Step 2 is the main concern of the subsequent sections of this paper. We know that the number of non-orientable embeddings of a graph G grows faster than the number of orientable embeddings, by the exponential factor $2^{\beta(G)}$, where $\beta(G)$ is the Betti number, and that determination by pre-existing methods of the crosscap-number polynomial $\mathcal{X}_G(y)$ has typically involved more complicated detail than the genus polynomial $\Gamma_G(y)$. Nonetheless, we shall see that the Euler-genus polynomial $\mathcal{E}_G(y)$ can be calculated from a simultaneous recursion of partial Euler-genus polynomials. From [16], we know that the number of embeddings types can be quite large. We illustrate in some examples the surprising discovery that the number of embedding types for the Euler-genus polynomial simultaneous recursion need not exceed the number of types for the genus polynomial recursion.

3. Revisiting Ladder Graphs with Euler-Genus Polynomials

In this section, we demonstrate our Three-Step Procedure to calculate the crosscap-number polynomial of a graph via its Euler-genus. As a first example, we have chosen the classical ***ladder graph*** $L_n = P_{n+2} \square K_2$, the cartesian product of the complete graph K_2 with the path graph that has $n + 2$ vertices and $n + 1$ edges, with $n \geq 1$. The subscript n for L_n reflects the graphic representation in Figure 3.1, in which there are n vertical rungs with two 3-valent endpoints. The leftmost and rightmost edges $(v_1, u_1)(v_1, u_2)$ and $(v_{n+2}, u_1)(v_{n+2}, u_2)$ of L_n are called ***end-rungs*** [12].

This sequence of ladders was the first non-trivial sequence of graphs whose genus polynomial was known [12]. It has been recalculated by different methods in [9, 40, 45]. Using the overlap matrix, Chen, Gross and Rieper [3] obtained an explicit formula for the crosscap-number polynomial of the ladder graph.

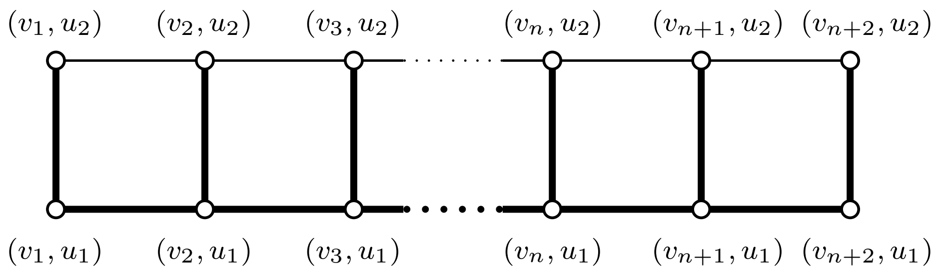

FIGURE 3.1. The ladder graph L_n

Step 1. The genus polynomial. The genus polynomial of L_n, adapted from [12], is given by

$$\Gamma_{L_n}(x) = \sum_{j=0}^{\lfloor (n+1)/2 \rfloor} \binom{n+1-j}{j} \frac{2n+2-3j}{n+1-j}\, 2^{n+j-1} x^j, \tag{3.1}$$

where it was calculated by partitioning the embeddings.

Step 2. The Euler-genus polynomial. As when calculating the genus distribution of the ladder graph L_n in [12], we partition the embeddings on a surface of Euler-genus i into two ***partial Euler-genus distributions***, as follows:

$\varepsilon^d_{L_n}(i)$**:** the number of embeddings of L_n on the surface of Euler-genus i such that the end-rung $(v_{n+2},u_1)(v_{n+2},u_2)$ lies on the boundaries of two different faces.

$\varepsilon^s_{L_n}(i)$**:** the number of embeddings of L_n on the surface of Euler-genus i such that the end-rung $(v_{n+2},u_1)(v_{n+2},u_2)$ lies twice on the boundary of the same face.

The two ***partial Euler-genus polynomials*** of L_n are

$$\begin{aligned}
\mathcal{E}^d_{L_n}(y) &= \sum_{i\ge 0} e^d_{L_n}(i) y^i, \text{ and}\\
\mathcal{E}^s_{L_n}(y) &= \sum_{i\ge 0} e^s_{L_n}(i) y^i, \text{ with}\\
\mathcal{E}_{L_n}(y) &= \mathcal{E}^d_{L_n}(y) + \mathcal{E}^s_{L_n}(y).
\end{aligned}$$

We shall now find a closed formula for $\mathcal{E}_{L_n}(y)$.

Theorem 3.1. *The Euler-genus polynomials of L_n satisfy the following second-order recurrence relation*

$$\mathcal{E}_{L_n}(y) = (2+4y)\mathcal{E}_{L_{n-1}}(y) + 16y^2\mathcal{E}_{L_{n-2}}(y) \tag{3.2}$$

with initial conditions $\mathcal{E}_{L_0}(y) = 1+y$ and $\mathcal{E}_{L_1}(y) = 2+6y+8y^2$.

Proof. We fix a spanning tree T of L_n, as indicated in Figure 3.1 by the thicker lines, i.e. the co-tree edges are

$$(v_1, u_2)(v_2, u_2),\ (v_2, u_2)(v_3, u_2),\ \ldots,\ (v_{n+1}, u_2)(v_{n+2}, u_2).$$

Note that the ladder L_n can be obtained from L_{n-1} by amalgamating the two end-vertices of a 4-vertex path-graph with the two vertices (v_{n+1}, u_1) and (v_{n+1}, u_2) of the end-rung in L_{n-1}. The proof has two cases.

Case 1: The end-rung $(v_{n+1}, u_1)(v_{n+1}, u_2)$ of L_{n-1} lies on the boundary of two different faces on the surface of Euler-genus i.

(1) If the cotree edge $(v_{n+1}, u_2)(v_{n+2}, u_2)$ is untwisted, then, via face-tracing and Euler's formula, we obtain 2 embeddings of L_n on surfaces of Euler-genus i, such that the end-rung lies on two different faces, and 2 embeddings of L_n on a surface of Euler-genus $i + 2$ such that the end-rung lies on a boundary of the same face.

(2) If the cotree edge $(v_{n+1}, u_2)(v_{n+2}, u_2)$ is twisted, then we obtain 2 embeddings of L_n on s surface of Euler-genus $i + 1$ such that the end-rung lies on the boundary of the same face, and 2 embeddings of L_n on a surface of Euler- genus $i + 2$ such that the end-rung lies on the boundary of the same face.

Case 2: The endrung $(v_{n+1}, u_1)(v_{n+1}, u_2)$ of L_{n-1} lies twice on the boundary of the same face on the surface of Euler-genus i.

(1) If the cotree edge $(v_{n+1}, u_2)(v_{n+2}, u_2)$ is untwisted, then we obtain 4 embeddings of L_n on a surface of Euler-genus i such that the end-rung lies on two different faces.

(2) If the cotree edge $(v_{n+1}, u_2)\ (v_{n+2}, u_2)$ is twisted, we obtain instead 4 embeddings of L_n on a surface of Euler-genus $i + 1$ such that the end-rung lies on the boundary of the same face.

Figures 3.2 and 3.3 illustrate Cases 1 and 2, respectively.

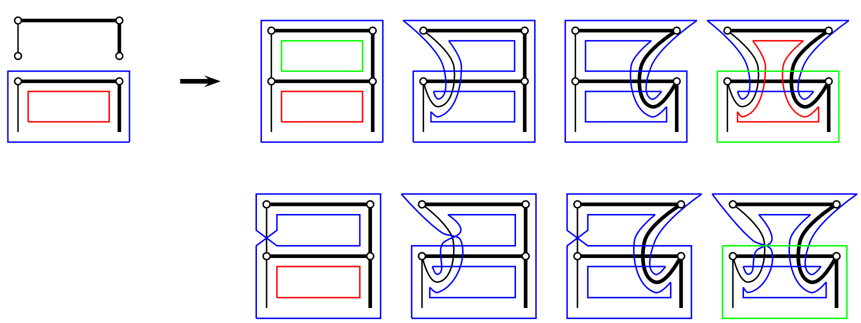

FIGURE 3.2. Case 1 of adding an end-rung to an embedding of L_{n-1}.

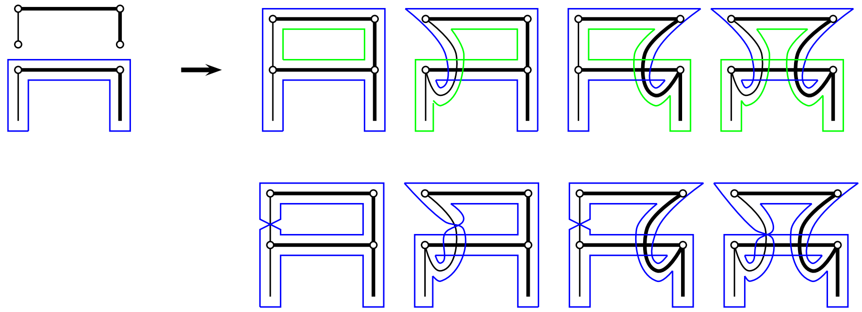

FIGURE 3.3. Case 2 of adding an end-rung to an embedding of L_{n-1}.

We see from Figure 3.2 that among the eight possible outcomes of adding a new end-rung two to a Case 1 embedding of L_{n-1}, exactly two (top row, 1^{st} and 4^{th}) embeddings of L_n have one additional face and remain in Case 1, and are therefore in a surface of the same Euler-genus. In Figure 3.3, exactly four possible outcomes (entire top row) have one additional face and switch over into Case 1. This is expressed algebraically by Recursion (3.3).

We see also from Figure 3.2 that exactly four of the embeddings of L_n that switch to Case 2 have one fewer faces, and are therefore in a surface of with Euler-genus increased by 2. Moreover, the two other consequent embeddings of L_n that switch to Case 2 have the same number of faces, which implies an Euler-genus increment of 1. In Figure 3.3, the four outcomes (entire bottom row) that remain in Case 2 have no additional faces, which again implies an Euler-genus increment of 1. This is expressed algebraically by Recursion (3.4).

$$e^d_{L_n}(i) = 2e^d_{L_{n-1}}(i) + 4e^s_{L_{n-1}}(i), \tag{3.3}$$

$$e^s_{L_n}(i) = 2e^d_{L_{n-1}}(i-1) + 4e^d_{L_{n-1}}(i-2) + 4e^s_{L_{n-1}}(i-1). \tag{3.4}$$

Multiplying Equations (3.3) and (3.4) by y^i and then summing leads to the following simultaneous recurrence system for partial Euler-genus polynomials of the ladder graph L_n:

$$\mathcal{E}^d_{L_n}(y) = 2\mathcal{E}^d_{L_{n-1}}(y) + 4\mathcal{E}^s_{L_{n-1}}(y) \tag{3.5}$$

$$\mathcal{E}^s_{L_n}(y) = (2y + 4y^2)\mathcal{E}^d_{L_{n-1}}(y) + 4y\mathcal{E}^s_{L_{n-1}}(y) \tag{3.6}$$

with initial conditions $\mathcal{E}^d_{L_0}(y) = 1$ and $\mathcal{E}^s_{L_0}(y) = y$. From equations (3.5) and (3.6), we obtain

$$\begin{aligned}
\mathcal{E}_{L_n}(y) &= \mathcal{E}^d_{L_n}(y) + \mathcal{E}^s_n(y) \\
&= (2+2y+4y^2)\mathcal{E}^d_{L_{n-1}}(y) + (4+4y)\mathcal{E}^s_{L_{n-1}}(y) \\
&= (2+4y)\left(\mathcal{E}^d_{L_{n-1}}(y) + \mathcal{E}^s_{L_{n-1}}(y)\right) + (4y^2-2y)\mathcal{E}^d_{L_{n-1}}(y) + 2\mathcal{E}^s_{L_{n-1}}(y) \\
&= (2+4y)\left(\mathcal{E}^d_{L_{n-1}}(y) + \mathcal{E}^s_{L_{n-1}}(y)\right) + 16y^2\left(\mathcal{E}^d_{L_{n-2}}(y) + \mathcal{E}^s_{L_{n-2}}(y)\right) \\
&= (2+4y)\mathcal{E}_{L_{n-1}}(y) + 16y^2\mathcal{E}_{L_{n-2}}(y).
\end{aligned}$$

The result follows. $\square$

We recall that the n^{th} ***Chebyshev polynomial*** $U_n(x)$ ***of the second kind*** is defined as follows:

$$U_n(x) = 2xU_{n-1}(x) - U_{n-2}(x)$$

with the initial values $U_0(x) = 1$ and $U_1(x) = 2x$. Moreover, the explicit formula for $U_n(x)$ is given by

$$U_n(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k}{k}(-1)^k(2x)^{n-2k}. \tag{3.7}$$

We now find an explicit formula for the Euler-genus polynomial of the ladder graph.

Theorem 3.2. *For all $n \geq 2$,*

$$\mathcal{E}_{L_n}(y) = (4iy)^n\left[\frac{1+3y+4y^2}{1+2y}U_n\left(\frac{1+2y}{4iy}\right) + \frac{2y^2}{1+2y}U_{n-2}\left(\frac{1+2y}{4iy}\right)\right],$$

where $U_s(t)$ is the s^{th} Chebyshev polynomial of the second kind and $i^2 = -1$.

Proof. We define $L(y,t) = \sum_{i\geq 0}\mathcal{E}_{L_i}(y)t^i$. Then, by Equation (3.2), the generating function $L(y,t)$ for the sequence of Euler-genus polynomials $\mathcal{E}_{L_n}(y)$ is given by the formula

$$-\frac{4\,ty^2+y+1}{16\,t^2y^2+4\,ty+2\,t-1}.$$

For $n \geq 2$, the coefficient of t^n in $L(y,t)$ is given by

$$\begin{aligned}
\mathcal{E}_{L_n}(y) &= [t^n]L(y,t) \\
&= (4iy)^n\left[\frac{1+3y+4y^2}{1+2y}U_n\left(\frac{1+2y}{4iy}\right) + \frac{2y^2}{1+2y}U_{n-2}\left(\frac{1+2y}{4iy}\right)\right], \qquad (3.8)
\end{aligned}$$

which completes the proof. $\square$

Step 3. The crosscap-number polynomial. From Theorem 2.1, Formula (3.1), and Theorem 3.2, we have the following result.

Theorem 3.3. [3, 9] *For $n \geq 2$, the crosscap-number polynomial of the ladder graph L_n is as follows:*

$$\begin{aligned} \mathcal{X}_{L_n}(y) &= (4iy)^n \left[\frac{1+3y+4y^2}{1+2y} U_n\left(\frac{1+2y}{4iy}\right) + \frac{2y^2}{1+2y} U_{n-2}\left(\frac{1+2y}{4iy}\right) \right] \\ &\quad - \sum_{j=0}^{\lfloor (n+1)/2 \rfloor} \binom{n+1-j}{j} \frac{2n+2-3j}{n+1-j} 2^{n+j-1} y^{2j}. \qquad \square \end{aligned}$$

Tables 3.1 and 3.2 contains some values of $\mathcal{E}_{L_n}(y)$ and $\mathcal{X}_{L_n}(y)$, for the smallest few ladders.

TABLE 3.1. Euler-genus polynomials for the smallest few ladders.

n	$\mathcal{E}_{L_n}(y)$
$n = 0$	$1 + y$
$n = 1$	$2(1 + 3y + 4y^2)$
$n = 2$	$4(1 + 5y + 14y^2 + 12y^3)$
$n = 3$	$8(1 + 7y + 28y^2 + 52y^3 + 40y^4)$
$n = 4$	$16(1 + 9y + 46y^2 + 128y^3 + 200y^4 + 128y^5)$
$n = 5$	$32(1 + 11y + 68y^2 + 248y^3 + 568y^4 + 736y^5 + 416y^6)$
$n = 6$	$64(1 + 13y + 94y^2 + 420y^3 + 1248y^4 + 2384y^5 + 2688y^6 + 1344y^7)$

TABLE 3.2. Crossing-number polynomials for the smallest few ladders.

n	$\mathcal{X}_{L_n}(y)$
$n = 0$	y
$n = 1$	$2y(3 + 3y)$
$n = 2$	$4y(5 + 11y + 12y^2)$
$n = 3$	$8y(7 + 23y + 52y^2 + 38y^3)$
$n = 4$	$16y(9 + 39y + 128y^2 + 192y^3 + 128y^4)$
$n = 5$	$32y(11 + 59y + 248y^2 + 550y^3 + 736y^4 + 412y^5)$
$n = 6$	$64y(13 + 83y + 420y^2 + 1216y^3 + 2384y^4 + 2668y^5 + 1344y^6)$

REMARK Similarly, we can also find Euler-genus polynomials for the cobblestone path graphs, Ringel ladders, circular ladders, and other graph sequences. However these calculations are considerably longer, and we would not adopt the method above. Recently, Chen, Gross and Mansour [7, 5] found a new way to calculate genus distributions for these graph families. We observe that the recent results of [5] can be extended to Euler-genus distributions. It follows that we could calculate Euler-genus distributions for these graph families within several pages, without recourse to overlap matrices.

4. Euler-Genus polynomials for H-spider-like families

In this section, we give the theorems needed to apply the Three-Step Procedure of Section 2 to *H-linear families*, in a form that permits them to be applied also to *ring-like families* and, more generally, to *spider-like families*. We then apply them to the calculation of crosscap-number polynomials for sequences of *iterated claws* and *grid-graphs*, both of which are H-linear sequences. We then apply the theorems to Möbius ladders, which are ring-like, and to a spider-like variation on ladders.

4.1. **H-spider-like families.** The concept of an H-linear family of graphs was defined by Stahl [40], for studying the genus distributions of graphs. The present authors [5] have recently extended the concept of H-linearity to *H-spider-like linearity*, which includes the oft-studied case of *ring-like graph families*.

Let H be a graph. Let $U_1 = \{u_1, u_2, \ldots, u_s\}$ and $V_1 = \{v_1, v_2, \ldots, v_s\}$ be two disjoint subsets of $V(H)$. For each $i \geq 1$, let H_i be a copy of H, and let $f_i : H_i \to H$ be an isomorphism. For each $i \geq 1$ and $1 \leq j \leq s$, let

$$U_{i,1} = \{u_{i,1}, u_{i,2}, \ldots, u_{i,s}\} \text{ and } V_{i,1} = \{v_{i,1}, v_{i,2}, \ldots, v_{i,s}\}$$

be two disjoint subsets of $V(H_i)$. A family of graphs $\{G_n\}_{n=1}^{\infty}$ is said to be ***H-linear*** if G_n is obtained from $\{H_1, H_2, \ldots, H_n\}$, the set of copies of H, by amalgamating each vertex $v_{i,j}$ of H_i with the vertex $u_{i+1,j}$ of H_{i+1}, for each $i = 1, 2, \ldots, n-1$ and $j = 1, 2, \ldots, s$. In short, an H-linear family of graphs can be obtained by iteratively amalgamating copies of H. Figure 4.1 shows a graph G_n in a generic H-linear sequence. Figure 4.3 and Figure 4.4 illustrate the specific examples of iterated claws and grid-graphs.

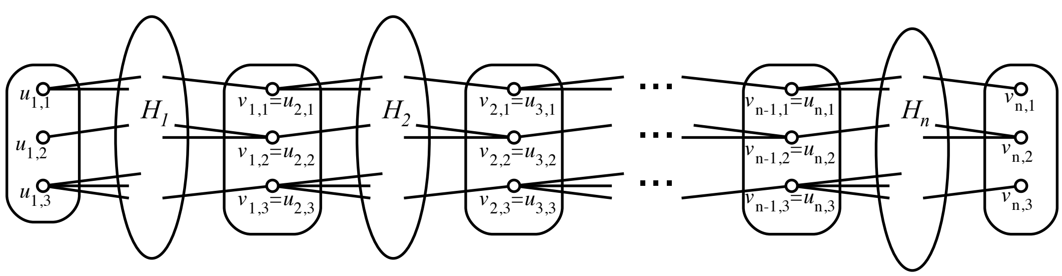

Figure 4.1. The graph G_n in a generic H-linear sequence.

Elsewhere we allow a "linear sequence" of copies of the iterated subgraph H to be amalgamated along single edges [35], or along equivalently embedded copies of arbitrary subgraphs [16], rather than requiring the amalgamations to be at sets of vertices. The difference is that, as illustrated in Figure 4.3 below, our present definition of an H-linear sequence may leave some "loose ends". Accordingly, the polynomials of interest for a linear sequence of graphs with loose ends at its extremities may require division by some scalar multiples to eliminate the algebraic contribution of the loose ends.

We let $\{G_n\}_{n=1}^{\infty}$ be an H-linear family of graphs, as defined above. Then for each $i \geq 1$, let (J_i, t_i) and $(\overline{J}_i, \overline{t_i})$ be graphs with root-vertices t_i and $\overline{t_i}$, respectively. We construct an ***H-spider-like graph*** G_n° from G_n by amalgamating the vertex $u_{1,j}$ of G_n with the vertex t_j of J_j and amalgamating the vertex $v_{1,j}$ of G_n with $\overline{t_j}$ of $\overline{J}_j$, for $j = 1, 2, \ldots, s$. The graphs (J_i, t_i) and $(\overline{J}_i, \overline{t}_i)$ that are amalgamated to the extreme copies of H in the chain are called ***attachments***. Figure 4.2 shows a generic example of a spider-like sequence. As a concrete example of a spider-like sequence, Figure 4.5 in Subsection 4.5 illustrates a variation on the ladder graphs..

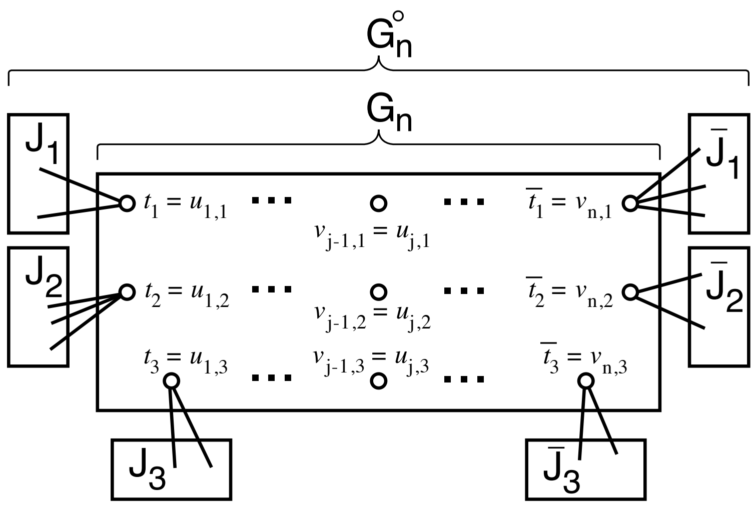

FIGURE 4.2. The H-spider-like graph G_n°.

The special case of an H-spider-like graph G_n° in which there exist $i, j \geq 1$ such that $J_i = \overline{J}_j$ is called ***H-ring-like***. It is easy to see that the necklaces [18], Ringel ladders [11, 43], and circular ladders [31, 6] are ring-like graph families.

Let $\mathbb{Z}[x]$ be the set of polynomials in x with integer coefficients. Recently, the present authors [5] have used the Cayley-Hamilton theorem to derive the following result on the genus distributions for H-spider-like families of graphs.

Theorem 4.1. [5] *Let $\mathcal{G} = \{G_n^\circ\}_{n=1}^\infty$ be an H-spider-like family of graphs, with attachments (J_i, t_i) and $(\overline{J}_i, \overline{t}_i)$, for $i = 1, \ldots, s$. Then there exist a positive integer k and polynomials $a_1(x), a_2(x), \ldots, a_k(x) \in \mathbb{Z}[x]$ such that the genus polynomial $\Gamma_{G_n^\circ}(x)$ satisfies the k^{th}-order homogeneous recurrence relation*

$$(4.1)\ \Gamma_{G_n^\circ}(x) \ = \ a_1(x)\Gamma_{G_{n-1}^\circ}(x) + a_2(x)\Gamma_{G_{n-2}^\circ}(x) + \cdots + a_k(x)\Gamma_{G_{n-k}^\circ}(x),$$

with $\Gamma_{G_1^\circ}(x)$, $\Gamma_{G_2^\circ}(x)$, $\ldots$, $\Gamma_{G_k^\circ}(x)$ as initial values. □

Let G_n° be the spider-like graph with root vertices $u_{1,1}, u_{1,2}, \ldots, u_{1,s}$ in H_1 and $v_{n,1}, v_{n,2}, \ldots, v_{n,s}$ in H_n. Suppose that there are k embedding types for the graph G_n°, labeled $1, 2, \ldots, k$. For $1 \leq j \leq k$, let $\varepsilon_{G_n}^j(i)$ be the number of embeddings of G_n° in S_i of embedding type j. We define ***partial Euler-genus polynomials*** of G_n° as generating functions

$$\mathcal{E}_{G_n}^j(y) = \sum_{i \geq 0} \varepsilon_{G_n}^j(i) y^i,$$

where $1 \leq j \leq k$. Using the same argument as in the proof of Theorem 2.3 of [5], while replacing the partial genus polynomial $\Gamma_{G_n}^j(y)$ by the partial Euler-genus polynomial $\mathcal{E}_{G_n}^j(y)$, for $1 \leq j \leq k$, we can easily carry out the proof of the following analogous theorem.

Theorem 4.2. *Let $\mathcal{G} = \{G_n^\circ\}_{n=1}^\infty$ be an H-spider-like family of graphs, with attachments (J_i, t_i) and $(\overline{J}_i, \overline{t}_i)$, for $i = 1, \ldots, s$. Then there exist a positive integer k and polynomials $b_1(y), b_2(y), \ldots, b_k(y) \in \mathbb{Z}[y]$, such that the Euler-genus polynomial $\mathcal{E}_{G_n^\circ}(y)$ satisfies the k^{th}-order homogeneous recurrence relation*

$$(4.2)\ \mathcal{E}_{G_n^\circ}(y) \ = \ b_1(y)\mathcal{E}_{G_{n-1}^\circ}(y) \ + \ b_2(y)\mathcal{E}_{G_{n-2}^\circ}(y) \ + \ \cdots \ + \ b_k(y)\mathcal{E}_{G_{n-k}^\circ}(y),$$

with initial conditions $\mathcal{E}_{G_1^\circ}(y)$, $\mathcal{E}_{G_2^\circ}(y)$, $\ldots$, $\mathcal{E}_{G_k^\circ}(y)$. □

Remark From the proof of [5, Theorem 2.3], we see that if the number of partial Euler-genus polynomials of G_n° is k (i.e., the embeddings of G_n° can be partitioned into k embedding types), then there is a set of l simultaneous recursions for $E_{G_n^\circ}(y)$, where $l \leq k$.

In Subsections 4.2 and 4.3, we shall use Theorem 4.2 to calculate the Euler-genus polynomials and then the crosscap polynomials for two H-linear graph sequences whose crosscap distributions have not previously been known, namely, the iterated claw graphs of [17] and the grid graphs of [26]. In Subsections 4.4 and 4.5, the same procedure for Euler-genus will be applied to the ring-like sequence of Möbius ladders and to an H-spider-like variation on the ladder sequences, thereby demonstrating the application of Theorem 4.2 in full generality.

4.2. **Euler-genus polynomial for the iterated-claw graph.** Let Y_n be the ***iterated-claw graph*** of Figure 4.3. The genus polynomial of Y_n has been obtained in [17]. Here we calculate the Euler-genus distribution for Y_n.

Theorem 4.3. *The Euler-genus polynomial for the iterated-claw graph Y_n is given by*

$$\begin{aligned}(4.3)\quad \mathcal{E}_{Y_n}(y) &= 2(3y+28y^2)\mathcal{E}_{Y_{n-1}}(y) - 16(-3y^2-12y^3+4y^4)\mathcal{E}_{Y_{n-2}}(y)\\ &\quad -3072y^6\mathcal{E}_{Y_{n-3}}(y),\end{aligned}$$

with initial values

$$\begin{aligned}\mathcal{E}_{Y_1}(y) &= 2,\\ \mathcal{E}_{Y_2}(y) &= 8(2+6y+8y^2), \quad \textit{and}\\ \mathcal{E}_{Y_3}(y) &= 32y(3+37y+108y^2+108y^3).\end{aligned}$$

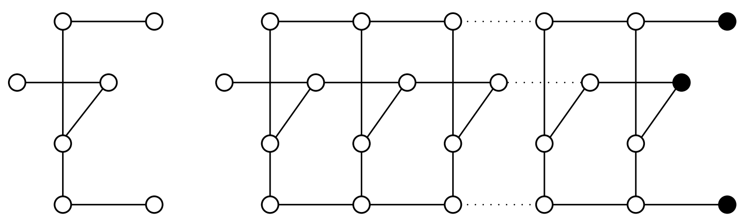

FIGURE 4.3. The claw Y (left), and the iterated-claw graph Y_n (right).

Proof. We choose the rightmost vertices of Y_n as its roots, as shown in black vertices of Figure 4.3. With symmetry, the embedding types of Y_n can be can be partitioned into 3 types (see [17] for details). From Theorem 4.2, we can suppose that the Euler-genus polynomials of Y_n satisfy the following third-order linear recursive equation

$$(4.4)\qquad \mathcal{E}_{Y_n}(y) = b_1(y)\mathcal{E}_{Y_{n-1}}(y) + b_2(y)\mathcal{E}_{Y_{n-2}}(y) + b_3(y)\mathcal{E}_{Y_{n-3}}(y).$$

With the help of a computer program, we get

$$\begin{aligned}(4.5)\qquad \mathcal{E}_{Y_1}(y) &= 2\\ (4.6)\qquad \mathcal{E}_{Y_2}(y) &= 8(2+6y+8y^2)\\ (4.7)\qquad \mathcal{E}_{Y_3}(y) &= 32y(3+37y+108y^2+108y^3)\\ (4.8)\qquad \mathcal{E}_{Y_4}(y) &= 64y^2(21+279y+1536y^2+3492y^3+2864y^4)\\ (4.9)\qquad \mathcal{E}_{Y_5}(y) &= 128y^3(99+2013y+15444y^2+58988y^3\\ &\quad +108672y^4+76928y^5)\\ (4.10)\qquad \mathcal{E}_{Y_6}(y) &= 256y^4(549+13167y+134184y^2+719412y^3\\ &\quad +2140880y^4+3313728y^5+2066688y^6).\end{aligned}$$

By Recursion (4.4), together with Formulas (4.5), (4.6), (4.7), (4.8), (4.9) and (4.10), we can construct the following system of three linear equations in the three unknowns $b_1(y), b_2(y), b_3(y)$.

$$\begin{aligned}
\mathcal{E}_{Y_4}(y) &= b_1(y)\mathcal{E}_{Y_3}(y) + b_2(y)\mathcal{E}_{Y_2}(y) + b_3(y)\mathcal{E}_{Y_1}(y) \\
\mathcal{E}_{Y_5}(y) &= b_1(y)\mathcal{E}_{Y_4}(y) + b_2(y)\mathcal{E}_{Y_3}(y) + b_3(y)\mathcal{E}_{Y_2}(y) \\
\mathcal{E}_{Y_6}(y) &= b_1(y)\mathcal{E}_{Y_5}(y) + b_2(y)\mathcal{E}_{Y_4}(y) + b_3(y)\mathcal{E}_{Y_3}(y)
\end{aligned} \tag{4.11}$$

Since the corresponding square coefficient matrix has a nonzero determinant, the linear system (4.11) has a unique solution (e.g., via *Cramer's rule*)

$$\begin{aligned}
b_1(y) &= 2(3y + 28y^2) \\
b_2(y) &= -16(-3y^2 - 12y^3 + 4y^4) \\
b_3(y) &= -3072y^6.
\end{aligned}$$

It follows that the Euler-genus polynomial of the iterated-claw graph Y_n is given by the recursion

$$\begin{aligned}
\mathcal{E}_{Y_n}(y) &= 2(3y + 28y^2)\mathcal{E}_{Y_{n-1}}(y) - 16(-3y^2 - 12y^3 + 4y^4)\mathcal{E}_{Y_{n-2}}(y) \\
&\quad - 3072y^6\mathcal{E}_{Y_{n-3}}(y),
\end{aligned} \tag{4.12}$$

with initial values

$$\begin{aligned}
\mathcal{E}_{Y_1}(y) &= 2, \\
\mathcal{E}_{Y_2}(y) &= 8(2 + 6y + 8y^2), \quad \text{and} \\
\mathcal{E}_{Y_3}(y) &= 32y(3 + 37y + 108y^2 + 108y^3).
\end{aligned}$$

For $1 \le l \le 3$, let

$$d_{n-3,l} = \sum_{t_1+2t_2+3t_3=n-4+l} \frac{t_l + t_{l+1} + \cdots + t_3}{t_1 + t_2 + t_3} \binom{t_1 + t_2 + t_3}{t_1, t_2, t_3} \left[\prod_{i=1}^{3} a_i(y)^{t_i}\right]. \tag{4.13}$$

Then, by applying the method of Mallik [32, §3.1.1], we derive the following explicit formula for $\mathcal{E}_{Y_n}(y)$, which was previously unknown:

$$\begin{aligned}
\mathcal{E}_{Y_n}(y) &= 32y(3 + 37y + 108y^2 + 108y^3)\, d_{n-3,1} \\
&\quad + 8(2 + 6y + 8y^2)\, d_{n-3,2} + 2d_{n-3,3}. \qquad \square
\end{aligned} \tag{4.14}$$

Combining our initial values for Euler-genus polynomials of iterated claws with the genus polynomials for iterated claws obtained in [17], we obtain the values for the crosscap-number polynomials of iterated-claw graphs listed in Table 4.1.

TABLE 4.1. Crosscap-number polynomials for a few iterated claws.

n	$\mathcal{X}_{Y_n}(y)$
$n=1$	0
$n=2$	$48y(1+y)$
$n=3$	$96y(1+9y+36y^2+34y^3)$
$n=4$	$192y^2(5+93y+482y^2+1164y^3+944y^4)$
$n=5$	$384y^3(33+631y+5148y^2+19420y^3+36224y^4+25584y^5)$
$n=6$	$768y^4(171+4389y+44180y^2+239804y^3+711776y^4$ $+1104576y^5+688576y^6)$

4.3. **Euler-genus polynomial for the grid graph.** Let $G_n = P_3 \square P_n$ be the ***grid graph***, as shown in Figure 4.4. In [26], Khan, Poshni and Gross obtained the genus distributions for this graph sequence G_n. Here we calculate the Euler-genus distribution for G_n.

Theorem 4.4. *The Euler-genus polynomial for the grid graph G_n is given by*

$$\begin{aligned} \mathcal{E}_{G_n}(y) &= (1+11y+84y^2)\mathcal{E}_{G_{n-1}}(y) + 12y^2(7+30y-28y^2)\mathcal{E}_{G_{n-2}}(y) \\ &\quad - 288y^4(1+4y+32y^2)\mathcal{E}_{G_{n-3}}(y) + 27648y^8\mathcal{E}_{G_{n-4}}(y), \end{aligned} \tag{4.15}$$

with initial values

$$\begin{aligned} \mathcal{E}_{G_1}(y) &= 2, \\ \mathcal{E}_{G_2}(y) &= 24(1+3y+4y^2), \\ \mathcal{E}_{G_3}(y) &= 24(1+14y+117y^2+320y^3+316y^4), \quad \textit{and} \\ \mathcal{E}_{G_4}(y) &= 24(1+25y+439y^2+3395y^3+14744y^4+30692y^5+24432y^6). \end{aligned}$$

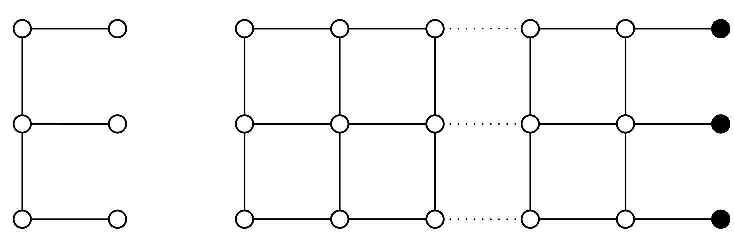

FIGURE 4.4. The grid graph G_n.

Proof. We proceed the same way as for iterated-claw graphs. We again use the method of Mallik to find an explicit formula for $\mathcal{E}_{G_n}(y)$:

$$\begin{aligned}\mathcal{E}_{G_n}(y) &= 24(1+25y+439y^2+3395y^3+14744y^4+30692y^5+24432y^6)e_{n-4,1}\\ &\quad +24(1+14y+117y^2+320y^3+316y^4)e_{n-4,2},\\ &\quad +24(1+3y+4y^2)e_{n-4,3}+2e_{n-4,4},\end{aligned}$$

where we define

$$e_{n-4,l} = \sum_{t_1+2t_2+3t_3+4t_4=n-5+l} \frac{t_l+t_{l+1}+\cdots+t_4}{t_1+t_2+t_3+t_4}\binom{t_1+t_2+t_3+t_4}{t_1,t_2,t_3,t_4}\left[\prod_{i=1}^{4} b_i(y)^{t_i}\right]. \tag{4.16}$$

with

$$\begin{aligned}b_1(y) &= 1+11y+84y^2,\\ b_2(y) &= 12y^2(7+30y-28y^2),\\ b_3(y) &= -288y^4(1+4y+32y^2),\\ b_4(y) &= 27648y^8.\end{aligned}$$

□

Similarly, we list some small values for the crosscap-number polynomials of the grid graph G_n, for $n \leq 6$.

Table 4.2. Crosscap-number polynomials for a few grid graphs.

n	$\mathcal{X}_{G_n}(y)$
$n=1$	0
$n=2$	$24y(3+3y)$
$n=3$	$24y(14+88y+320y^2+298y^3)$
$n=4$	$24y(25+338y+3395y^2+13982y^3+30692y^4+24144y^5)$
$n=5$	$24y(36+709y+11860y^2+98439y^3+540336y^4+1671196y^5$ $+2820864y^6+1906800y^7)$
$n=6$	$24y(47+1201y+27046y^2+339047y^3+3101511y^4+18383560y^5$ $+73178996y^6+181068048y^7+253232448y^8+149481792y^9)$

4.4. **Euler-genus polynomial for the Möbius ladders (ring-like).** The ***Möbius ladder*** ML_n is a cubic graph with an even number $2n$ of vertices, formed from an $2n$-cycle by adding edges connecting opposite pairs of vertices in the cycle. Just as the Möbius band can be described as an annulus with a half-twist, the Möbius ladder can be described as a circular ladder with a half-twist.

The Euler-genus polynomial for the Möbius ladder ML_n is given by the recursion

$$
\begin{aligned}
(4.17)\ \mathcal{E}_{ML_n}(y) \;=\;& (12y+4)\mathcal{E}_{ML_{n-1}}(y) + (-12y^2-34y-5)\mathcal{E}_{ML_{n-2}}(y) \\
&+ (-240y^3-20y^2+26y+2)\mathcal{E}_{ML_{n-3}}(y) \\
&+ 4(80y^3+128y^2+14y-1)y\mathcal{E}_{ML_{n-4}}(y) \\
&+ 16(112y^3+8y^2-14y-1)y^2\mathcal{E}_{ML_{n-5}}(y) \\
&- 128(8y^2+18y+3)y^4\mathcal{E}_{ML_{n-6}}(y) \\
&- 2048(2y+1)y^6\mathcal{E}_{ML_{n-7}}(y).
\end{aligned}
$$

with initial values

$$
\begin{aligned}
\mathcal{E}_{ML_1}(y) &= 6y^2+8y+2, \\
\mathcal{E}_{ML_2}(y) &= 2+14y+56y^2+56y^3, \\
\mathcal{E}_{ML_3}(y) &= 12y+148y^2+432y^3+432y^4, \\
\mathcal{E}_{ML_4}(y) &= 10y+214y^2+1272y^3+3496y^4+3200y^5, \\
\mathcal{E}_{ML_5}(y) &= 12y+292y^2+2480y^3+11872y^4+27264y^5+23616y^6, \\
\mathcal{E}_{ML_6}(y) &= 14y+438y^2+4540y^3+28528y^4+106032y^5 \\
&\quad +214752y^6+169984y^7, \quad \text{and} \\
\mathcal{E}_{ML_7}(y) &= 16y+660y^2+8260y^3+62440y^4+302288y^5 \\
&\quad +943840y^6+1664128y^7+1212672y^8.
\end{aligned}
$$

Defining

$$
M_{\mathcal{E}}(y,t) = \sum_{i\geq 3} \mathcal{E}_{ML_i}(y)t^i \tag{4.18}
$$

and then rewriting $M_{\mathcal{E}}(y,t)$ by partial fraction decomposition, we obtain the formula

$$
\begin{aligned}
M_{\mathcal{E}}(y,t) \;=\;& \left(-48\,y^3-56\,y^2-20\,y-4\right)t^2 + \left(-4\,y^2-8\,y-4\right)t \\
&+ \frac{2\,y^3-7\,y^2-2\,y-1}{4y^2} - \frac{y-1}{2\,ty+t-1} + \frac{6\,y^3+y^2-6\,y-1}{12\,(4\,ty-1)\,y^2} \\
&+ \frac{-6\,y^3-7\,y^2+1}{6\,(2\,ty+1)\,y^2} + \frac{4y^2\,(2\,ty+t-1)}{16\,t^2y^2+4\,ty+2\,t-1} \\
&+ \frac{-8\,ty^3+2\,ty^2+5\,ty+4\,y^2+t-2\,y-2}{16\,t^2y^2+4\,ty+t-1}
\end{aligned}
$$

Thus the coefficient $\mathcal{E}_{ML_n}(y)$ $(n\geq 3)$ of t^n in the generating function $M_{\mathcal{E}}(y,t)$ is given by the formula

$$
\begin{aligned}
(4.19)\ \mathcal{E}_{ML_n}(y) \ &= (y-1)(2y+1)^n - \frac{4^{n-1}}{3} y^{n-2}(y-1)(6y+1)(y+1) \\
&- \frac{2(-2y)^{n-2}(3y-1)(2y+1)(y+1)}{3} \\
&+ 2y^2(4yi)^n \left[U_n\left(\frac{1+2y}{4yi}\right) - U_{n-2}\left(\frac{1+2y}{4yi}\right)\right] \\
&+ (1+2y)(1-y)(4yi)^n \left[U_n\left(\frac{1+4y}{8yi}\right) - U_{n-2}\left(\frac{1+4y}{8yi}\right)\right],
\end{aligned}
$$

where the first, second, third and fourth lines are the coefficients of t^n in the generating functions

$$
\frac{1-y}{2\,ty+t-1} + \frac{6\,y^3+y^2-6\,y-1}{12\,(4\,ty-1)\,y^2}, \quad \frac{-6\,y^3-7\,y^2+1}{6\,(2\,ty+1)\,y^2}, \quad \frac{4y^2\,(2\,ty+t-1)}{16\,t^2y^2+4\,ty+2\,t-1}
$$

and $\dfrac{-8\,ty^3+2\,ty^2+5\,ty+4\,y^2+t-2\,y-2}{16\,t^2y^2+4\,ty+t-1}$, respectively.

Similarly, we list some small values for the crosscap-number polynomials of the Möbius ladder graph ML_n, for $n \le 6$.

TABLE 4.3. Crosscap-number polynomials for a few Möbius ladders.

n	$\mathcal{X}_{ML_n}(y)$
$n=1$	$8y+4y^2$
$n=2$	$14y+42y^2+56y^3$
$n=3$	$12y+108y^2+432y^3+408y^4$
$n=4$	$10y+158y^2+1272y^3+3296y^4+3200y^5$
$n=5$	$12y+220y^2+2480y^3+11240y^4+27264y^5+23296y^6$
$n=6$	$14y+326y^2+4540y^3+27200y^4+106032y^5+212096y^6+169984y^7$

4.5. **Euler-genus polynomial for a spider-like variation on ladders.** Let G_n be the P_4-spider-like graph of Figure 4.5. These spider-like graphs are neither H-linear, nor ring-like. Nonetheless, by proceeding as in previous examples, we obtain for the Euler-genus polynomial of G_n the recursion

$$
(4.20) \quad \mathcal{E}_{G_n}(y) \ = \ (2+4y)\mathcal{E}_{L_{n-1}}(y) + 16y^2\mathcal{E}_{G_{n-2}}(y)
$$

with the initial conditions

$$
\begin{aligned}
\mathcal{E}_{G_1}(y) \ &= \ 144+576y+864y^2+576y^3+144y^4, \\
\mathcal{E}_{G_2}(y)(z) \ &= \ 256+1536y+4288y^2+6336y^3+4672y^4+1344y^5.
\end{aligned}
$$

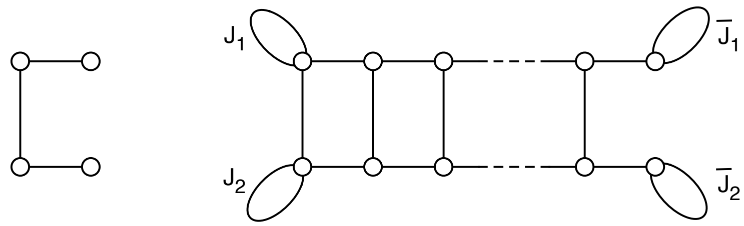

FIGURE 4.5. A ladder variation, with spider-attachments J_1, J_2, $\overline{J}_1$, $\overline{J}_2$.

By solving the recurrence system (4.20) above, we obtain

$$\begin{aligned}\frac{\mathcal{E}_{G_n}(y)}{(4iy)^{n-1}} &= \frac{32(1+y)^2(4+16y+31y^2+21y^3)}{1+2y}U_{n-1}\left(\frac{1+2y}{4iy}\right) \\ &\quad + \frac{16(1+y)^2(-1-4y+17y^2+24y^3)}{1+2y}U_{n-3}\left(\frac{1+2y}{4iy}\right),\end{aligned}$$

for all $n \geq 1$.

Similarly, we list some small values for the crosscap-number polynomials of the P_4-spider-like graph G_n for $n \leq 6$.

TABLE 4.4. Crosscap-number polynomials for a few P_4-spider-like graphs.

n	$\mathcal{X}_{G_n}(y)$
$n=1$	$144(-1+(1+y)^4)$
$n=2$	$64y(24+62y+99y^2+73y^3+21y^4)$
$n=3$	$128y(32+119y+305y^2+379y^3+239y^4+60y^5)$
$n=4$	$256y(40+191y+667y^2+1247y^3+1393y^4+830y^5+204y^6)$
$n=5$	$512y(48+279y+1221y^2+3085y^3+5127y^4+5132y^5$ $+2820y^6+648y^7)$
$n=6$	$1024y(56+383y+1999y^2+6335y^3+14041y^4+20394y^5$ $+18656y^6+9608y^7+2112y^8)$

5. FURTHER OBSERVATIONS

5.1. **bar-amalgamation.** A ***bar-amalgamation*** $G \oplus_{uv} H$ of two disjoint root graphs (G,u) and (H,v) is obtained by adding an edge between the vertex u of G and the vertex v of H. Theorem 5 of Gross and Furst [15] can be viewed as the first theorem on the genus polynomial (or distribution) of a graph. Theorem 5.2 below is an analogous result for the Euler-genus. We denote the degree of the vertex u in the graph G by $d_G(u)$.

Theorem 5.1. [15, Theorem 5]

$$\Gamma_{G\oplus_e H}(x) \;=\; d_G(u)d_H(v)\Gamma_G(x)\Gamma_H(x). \qquad \square$$

Here is the analogous result for the Euler-genus polynomial.

Theorem 5.2.

$$\mathcal{E}_{G\oplus_e H}(y) \;=\; d_G(u)d_H(v)\mathcal{E}_G(y)\mathcal{E}_H(y).$$

Proof. Let $(\rho_G, \lambda_{E(G)})$ and $(\rho_H, \lambda_{E(H)})$ be general rotation systems of the graphs G and H, respectively. Let $\rho_{G\oplus_e H}(u)$ be the rotation obtained by inserting the edge-end of uv at u somewhere into the rotation $\rho_G(u)$ of G at u. Let $\rho_{G\oplus_e H}(v)$ be the rotation obtained by inserting the other edge-end of uv somewhere into the rotation $\rho_H(v)$ of H at v. Then the resulting rotation system of $G \oplus_e H$ is

$$\rho_{G\oplus_e H} = \rho_{G\oplus_e H}(u) \cup \rho_{G\oplus_e H}(v) \cup \prod_{x\in V(G)-u} \rho_G(x) \cup \prod_{y\in V(H)-v} \rho_H(y),$$

with twist-indicator $\lambda = \lambda_{E(G)} \cup \lambda_{E(H)})$.

Suppose that the general rotation system $(\rho_G, \lambda_{E(G)})$ corresponds to an embedding of G into a surface with Euler-genus i , and that $(\rho_H, \lambda_{E(H)})$ corresponds to an embedding of H into a surface with Euler-genus j. The following four cases are considered.

Case 1: The general rotation system (ρ_G, λ_G) corresponds to an embedding of G on the surface $S_{\frac{i}{2}}$, and the general rotation system $(\rho_H, \lambda_{E(H)})$ corresponds to an embedding of H on the surface $S_{\frac{j}{2}}$. By face-tracing, the general rotation system $(\rho_{G\oplus_e H}, \lambda_{E(G)} \cup \lambda_{E(H)})$ corresponds to an embedding of $G \oplus_e H$ on the surface $S_{\frac{i+j}{2}}$.

Case 2: The general rotation system (ρ_G, λ_G) corresponds to an embedding of G on the surface $S_{\frac{i}{2}}$, and the general rotation system $(\rho_H, \lambda_{E(H)})$ corresponds to an embedding of H on the surface N_j. By face-tracing, the general rotation system $(\rho_{G\oplus_e H}, \lambda_{E(G)} \cup \lambda_{E(H)})$ corresponds to an embedding of $G \oplus_e H$ on the surface N_{i+j}.

Case 3: The general rotation system (ρ_G, λ_G) corresponds to an embedding of G on the surface N_i, and the general rotation system $(\rho_H, \lambda_{E(H)})$ corresponds to an embedding of H on the surface $S_{\frac{j}{2}}$. By face-tracing, the general rotation system $(\rho_{G\oplus_e H}, \lambda_{E(G)} \cup \lambda_{E(H)})$ corresponds to an embedding of $G \oplus_e H$ on the surface N_{i+j}.

Case 4: The general rotation system (ρ_G, λ_G) corresponds to an embedding of G on the surface N_i, and the general rotation system $(\rho_H, \lambda_{E(H)})$ corresponds to an embedding of H on the surface N_j. By face-tracing, the general rotation system $(\rho_{G\oplus_e H}, \lambda_{E(G)} \cup \lambda_{E(H)})$ corresponds to an embedding of $G \oplus_e H$ on the surface N_{i+j}. □

5.2. **Remarks about algorithms.** Thomassen [44] showed that deciding the minimum genus of a graph G is NP-complete. It follows that calculating the genus distribution of a graph is NP-hard. Kawarabayashi, Mohar, and Reed [24] presented a linear-time algorithm for the minimum genus of graphs of bounded tree-width. Gross [14] developed a quadratic-time algorithm to calculate the genus distributions of graphs with fixed tree-width and bounded degree.

We observe that with some minor adjustments, the specific genus distribution algorithms in [13, 19, 36] as well as the general algorithm [14] can be extended to Euler-genus distributions.

5.3. **Research Problems.** We conclude this paper with two research problems about the Euler-genus polynomials.

Problem 5.3. Kwak and Shim [28] calculated the crosscap-number distributions for bouquets of circles, with the aid of edge-attaching surgery technique. It is known that the genus distributions of bouquets of circles satisfy a second-order recurrence [21]. A natural problem is to find an analogous recurrence for the Euler-genus distribution of bouquets of circles.

Problem 5.4. A real sequence $a_0\, a_1, \ldots, a_n$ is called ***unimodal*** if for some number m such that $0 \le m \le n$, we have

$$a_0 \le a_1 \le \ldots \le a_m \ge a_{m+1} \ge \ldots \ge a_n,$$

in which case, m is called the mode of the sequence. Moreover, if for every j such that $1 \le j \le n-1$, we have ${a_j}^2 \ge a_{j-1}a_{j+1}$, then the sequence is called ***log-concave***. From Theorem 5.2, we can obtain an explicit formula for the Euler-genus distribution of a cactus (a graph with maxmum genus 0). One can easily prove that the Euler-genus distributions of the cacti are log-concave. Analogous to a conjecture [21], that the genus distribution of a graph is log-concave, we now ask whether the Euler-genus distribution of a graph is log-concave. We bear in mind the example of Auslander, Brown, and Youngs [1], of the existence of graphs of arbitrarily high minimum genus, whose minimum crosscap-number is 1.

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Department of Mathematics, Hunan University, 410082 Changsha, China
E-mail address: ycchen@hnu.edu.cn

Department of Computer Science, Columbia University, New York, NY 10027 USA
E-mail address: gross@cs.columbia.edu