Advanced Computer Graphics (Spring 2006)

COMS 4162, Lecture 3: Sampling and Reconstruction

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To Do

- Assignment 1, Due Feb 16.
  - Anyone need help finding partners?
  - Start thinking about written part based on this lecture

Outline

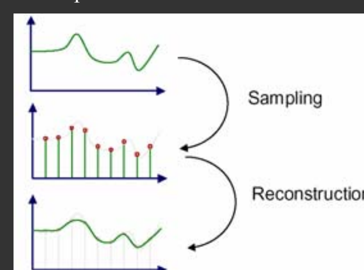
- *Basic ideas of sampling, reconstruction, aliasing*
- Signal processing and Fourier analysis
- Implementation of digital filters (second part of assn) next week

- Section 14.10 of textbook (you really should read it)

Some slides courtesy Tom Funkhouser

Sampling and Reconstruction

- An image is a 2D array of samples
- Discrete samples from real-world continuous signal



Sampling and Reconstruction

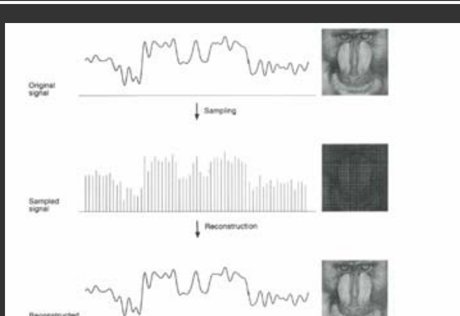
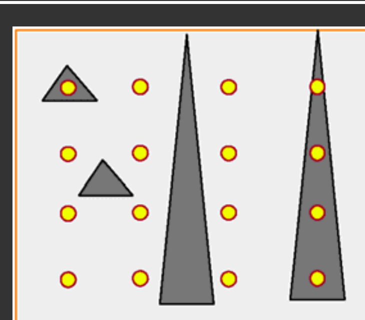


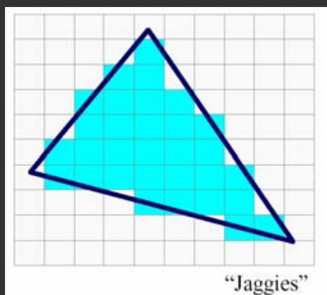
Figure 19.9 FvDFH

(Spatial) Aliasing



(Spatial) Aliasing

- Jaggies probably biggest aliasing problem



Sampling and Aliasing

- Artifacts due to undersampling or poor reconstruction
- Formally, high frequencies masquerading as low
- E.g. high frequency line as low freq jaggies

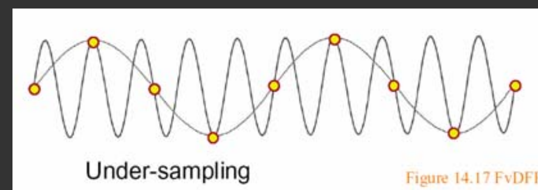
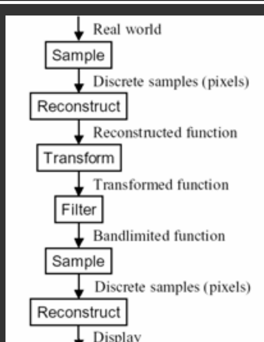


Image Processing pipeline



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Motivation

- Formal analysis of sampling and reconstruction
- Important theory (signal-processing) for graphics
- Mathematics tested in written assignment
- Will implement some ideas in project

Ideas

- Signal (function of time generally, here of space)
- Continuous: defined at all points; discrete: on a grid
- High frequency: rapid variation; Low Freq: slow variation
- Images are converting continuous to discrete. Do this sampling as best as possible.
- Signal processing theory tells us how best to do this
- Based on concept of frequency domain Fourier analysis

Sampling Theory

Analysis in the frequency (not spatial) domain

- Sum of sine waves, with possibly different offsets (phase)
- Each wave different frequency, amplitude

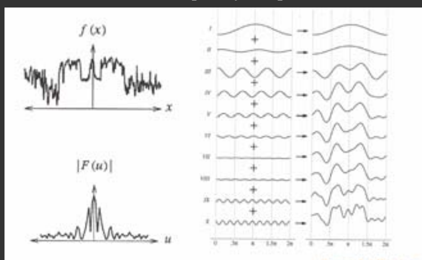


Figure 2.6 Wolberg

Fourier Transform

- Tool for converting from spatial to frequency domain
- Or vice versa
- One of most important mathematical ideas
- Computational algorithm: Fast Fourier Transform
  - One of 10 great algorithms scientific computing
  - Makes Fourier processing possible (images etc.)
  - Not discussed here, but see extra credit on assignment 1

Fourier Transform

- Simple case, function sum of sines, cosines

$$f(x) = \sum_{u=-\infty}^{+\infty} F(u) e^{2\pi iux}$$

$$F(u) = \int_0^{2\pi} f(x) e^{-2\pi iux} dx$$

- Continuous infinite case

Forward Transform: $F(u) = \int_{-\infty}^{\infty} f(x) e^{-2\pi iux} dx$

Inverse Transform: $f(x) = \int_{-\infty}^{+\infty} F(u) e^{2\pi iux} du$

Fourier Transform

- Simple case, function sum of sines, cosines

$$f(x) = \sum_{u=-\infty}^{+\infty} F(u) e^{2\pi iux}$$

$$F(u) = \int_0^{2\pi} f(x) e^{-2\pi iux} dx$$

- Discrete case

$$F(u) = \sum_{x=0}^{x=N-1} f(x)\left[\cos(2\pi ux/N) - i\sin(2\pi ux/n)\right], \qquad 0 \le u \le N-1$$

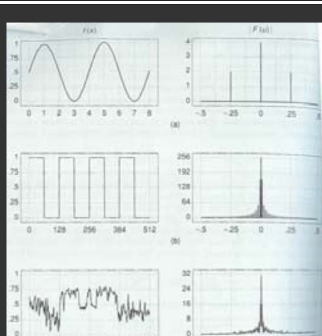
$$f(x) = \frac{1}{N} \sum_{u=0}^{u=N-1} F(u)\left[\cos(2\pi ux/N) + i\sin(2\pi ux/n)\right], \qquad 0 \le x \le N-1$$

Fourier Transform: Examples 1

Single sine curve
(+constant DC term)

$$f(x) = \sum_{u=-\infty}^{+\infty} F(u) e^{2\pi iux}$$

$$F(u) = \int_0^{2\pi} f(x) e^{-2\pi iux} dx$$



Fourier Transform Examples 2

Forward Transform: $F(u) = \int_{-\infty}^{\infty} f(x) e^{-2\pi iux} dx$

Inverse Transform: $f(x) = \int_{-\infty}^{+\infty} F(u) e^{2\pi iux} du$

- Common examples

$f(x)$	$F(u)$
$\delta(x - x_0)$	$e^{-2\pi iux_0}$
1	$\delta(u)$
e^{-ax^2}	$\sqrt{\pi/a}\, e^{-\pi^2 u^2 / a}$

Fourier Transform Properties

Forward Transform: $F(u) = \int_{-\infty}^{\infty} f(x) e^{-2\pi iux} dx$

Inverse Transform: $f(x) = \int_{-\infty}^{+\infty} F(u) e^{2\pi iux} du$

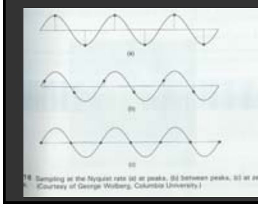
- Common properties
  - Linearity: $F(af(x) + bg(x)) = aF(f(x)) + bF(g(x))$
  - Derivatives: [integrate by parts] $F(f'(x)) = \int_{-\infty}^{\infty} f'(x) e^{-2\pi iux} dx$ $= 2\pi iuF(u)$
  - 2D Fourier Transform

    Forward Transform: $F(u,v) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y) e^{-2\pi iux} e^{-2\pi ivy} dxdy$

    Inverse Transform: $f(x,y) = \int_{-\infty}^{\infty} \int_{-\infty}^{+\infty} F(u,v) e^{2\pi iux} e^{2\pi ivy} dudv$
- Convolution (next)

Sampling Theorem, Bandlimiting

- A signal can be reconstructed from its samples, if the original signal has no frequencies above half the sampling frequency – Shannon
- The minimum sampling rate for a bandlimited function is called the Nyquist rate



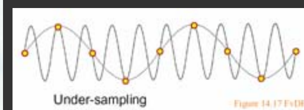


Figure 14.17 FvDFH

Sampling Theorem, Bandlimiting

- A signal can be reconstructed from its samples, if the original signal has no frequencies above half the sampling frequency – Shannon
- The minimum sampling rate for a bandlimited function is called the Nyquist rate
- A signal is bandlimited if the highest frequency is bounded. This frequency is called the bandwidth
- In general, when we transform, we want to filter to bandlimit before sampling, to avoid aliasing

Antialiasing

- Sample at higher rate
  - Not always possible
  - Real world: lines have infinitely high frequencies, can't sample at high enough resolution
- Prefilter to bandlimit signal
  - Low-pass filtering (blurring)
  - Trade blurriness for aliasing

Ideal bandlimiting filter

- Formal derivation is homework exercise

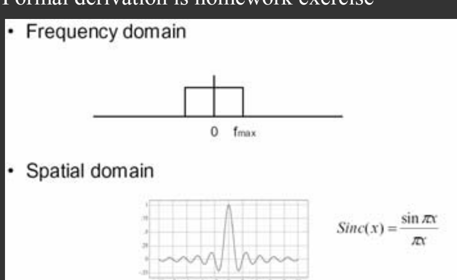


Figure 4.5 Wolberg

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  - *Convolution*
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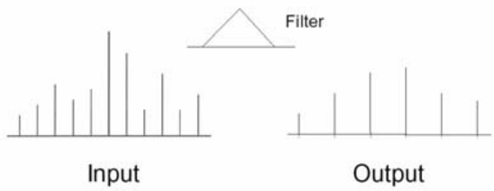
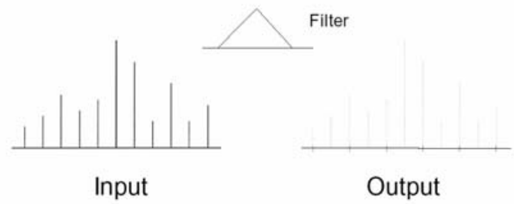
Convolution 1

- Spatial domain: output pixel is weighted sum of pixels in neighborhood of input image
  - Pattern of weights is the "filter"



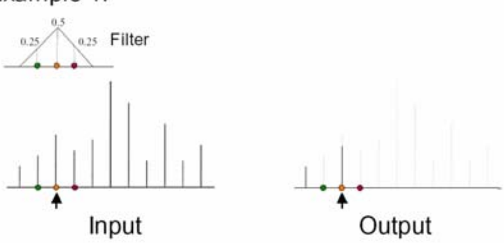
Convolution 2

- Example 1:



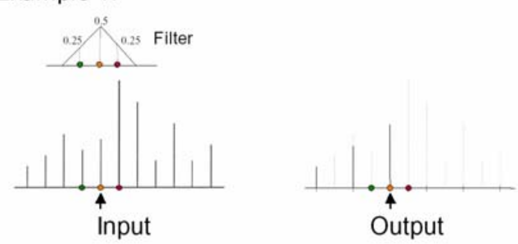
Convolution 3

- Example 1:



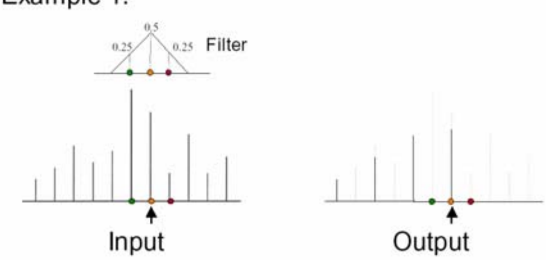
Convolution 4

- Example 1:



Convolution 5

- Example 1:



Convolution in Frequency Domain

Forward Transform: $F(u) = \int_{-\infty}^{\infty} f(x) e^{-2\pi iux} dx$

Inverse Transform: $f(x) = \int_{-\infty}^{+\infty} F(u) e^{2\pi iux} du$

- Convolution (f is signal ; g is filter [or vice versa])

$$h(y) = \int_{-\infty}^{+\infty} f(x) g(y-x) dx = \int_{-\infty}^{+\infty} g(x) f(y-x) dx$$

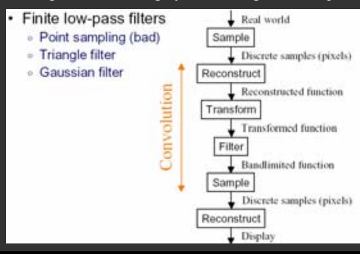
$$h = f * g \quad or \quad f \otimes g$$

- Fourier analysis (frequency domain multiplication)

$$H(u) = F(u) G(u)$$

Practical Image Processing

- Discrete convolution (in spatial domain) with filters for various digital signal processing operations
- Easy to analyze, understand effects in frequency domain
  - E.g. blurring or bandlimiting by convolving with low pass filter



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