Computer Graphics (Fall 2008)

COMS 4160, Lecture 6: Curves 1

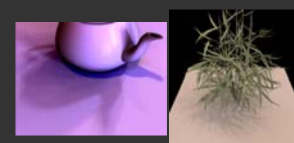
http://www.cs.columbia.edu/~cs4160

Course Outline

- 3D Graphics Pipeline

Modeling
(Creating 3D Geometry)

Rendering
(Creating, shading images from geometry, lighting, materials)



Course Outline

- 3D Graphics Pipeline

Modeling
(Creating 3D Geometry)

Rendering
(Creating, shading images from geometry, lighting, materials)

Unit 1: Transformations
Weeks 1,2. Ass 1 due Sep 25

Unit 2: Spline Curves
Modeling geometric objects
Weeks 3,4 hw2.exe
Ass 2 due Oct 7 (demo)

Motivation

- How do we model complex shapes?
  - In this course, only 2D curves, but can be used to create interesting 3D shapes by surface of revolution, lofting etc
- Techniques known as spline curves
- This unit is about mathematics required to draw these spline curves, as in HW 2
- History: From using computer modeling to define car bodies in auto-manufacturing. Pioneers are Pierre Bezier (Renault), de Casteljau (Citroen)

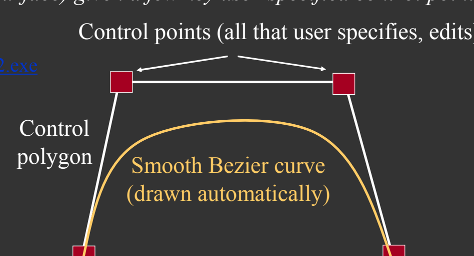
Outline of Unit

- Bezier curves
- deCasteljau algorithm, explicit form, matrix form
- Polar form labeling (next time)
- B-spline curves (next time)
- Not well covered in textbooks (especially as taught here). Main reference will be lecture notes. If you do want a printed ref, handouts from CAGD, Seidel

Bezier Curve (with HW2 demo)

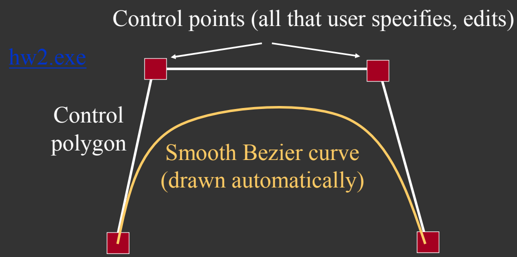
- *Motivation: Draw a smooth intuitive curve (or surface) given a few key user-specified control points*

hw2.exe



Bezier Curve: (Desirable) properties

- Interpolates, is tangent to end points
- Curve within convex hull of control polygon



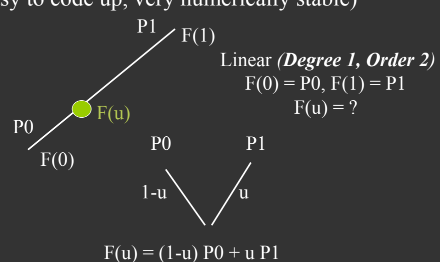
Issues for Bezier Curves

Main question: Given control points and constraints (interpolation, tangent), how to construct curve?

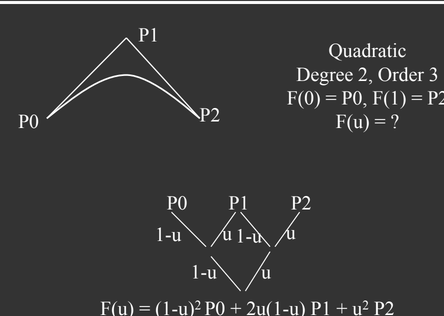
- *Algorithmic: deCasteljau algorithm*
- Explicit: Bernstein-Bezier polynomial basis
- 4x4 matrix for cubics
- Properties: Advantages and Disadvantages

deCasteljau: Linear Bezier Curve

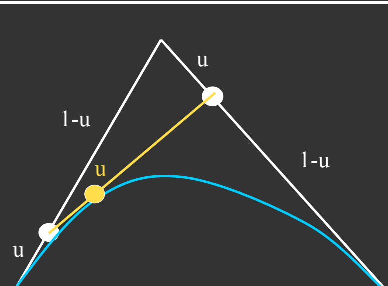
- Just a simple linear combination or interpolation (easy to code up, very numerically stable)



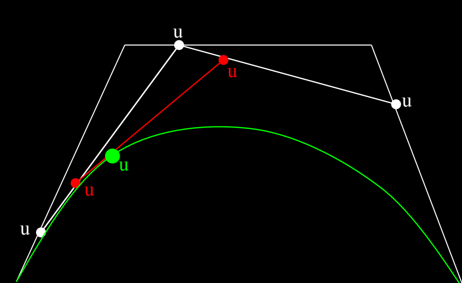
deCasteljau: Quadratic Bezier Curve



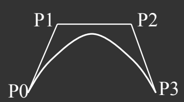
Geometric interpretation: Quadratic



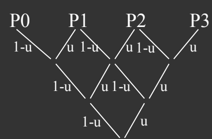
Geometric Interpretation: Cubic



deCasteljau: Cubic Bezier Curve

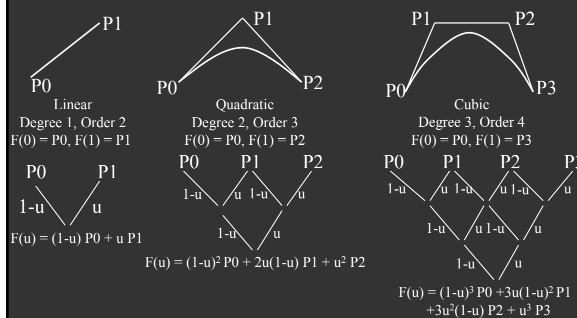


Cubic
Degree 3, Order 4
F(0) = P0, F(1) = P3



$F(u) = (1-u)^3 \, P0 + 3u(1-u)^2 \, P1 + 3u^2(1-u) \, P2 + u^3 \, P3$

Summary: deCasteljau Algorithm



DeCasteljau Implementation

Input: Control points C_i with $0 \le i \le n$ where n is the degree.
Output: $f(u)$ where u is the parameter for evaluation

for ($level = n$; $level \ge 0$; $level --$) {
  if ($level == n$) { // Initial control points
    $\forall i : 0 \le i \le n : p_i^{level} = C_i$; continue ; }
  for ($i = 0$; $i \le level$; $i++$)
    $p_i^{level} = (1-u) * p_i^{level+1} + u * p_{i+1}^{level+1}$;
}
$f(u) = p_0^0$

- Can be optimized to do without auxiliary storage

Summary of HW2 Implementation

Bezier (Bezier2 and Bspline discussed next time)
- Arbitrary degree curve (number of control points)
- Break curve into detail segments. Line segments for these
- Evaluate curve at locations 0, 1/detail, 2/detail, … , 1
- Evaluation done using deCasteljau

- Key implementation: deCasteljau for arbitrary degree
  - Is anyone confused? About handling arbitrary degree?
- Can also use alternative formula if you want
  - Explicit Bernstein-Bezier polynomial form (next)
- Questions?

Issues for Bezier Curves

Main question: Given control points and constraints (interpolation, tangent), how to construct curve?

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- *Explicit: Bernstein-Bezier polynomial basis*
- 4x4 matrix for cubics
- Properties: Advantages and Disadvantages

Recap formulae

- Linear combination of basis functions

Linear: $F(u) = P_0(1-u) + P_1 u$

Quadratic: $F(u) = P_0(1-u)^2 + P_1[2u(1-u)] + P_2 u^2$

Cubic: $F(u) = P_0(1-u)^3 + P_1[3u(1-u)^2] + P_2[3u^2(1-u)] + P_3 u^3$

Degree n: $F(u) = \sum_k P_k B_k^n(u)$ $\quad B_k^n(u)$ are Bernstein-Bezier polynomials

- Explicit form for basis functions? Guess it?

Recap formulae

- Linear combination of basis functions

Linear: $F(u) = P_0(1-u) + P_1 u$

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Degree n: $F(u) = \sum_k P_k B_k^n(u)$ $B_k^n(u)$ are Bernstein-Bezier polynomials

- Explicit form for basis functions? Guess it?
- ***Binomial coefficients in [(1-u)+u]^n***

Summary of Explicit Form

Linear: $F(u) = P_0(1-u) + P_1 u$

Quadratic: $F(u) = P_0(1-u)^2 + P_1[2u(1-u)] + P_2 u^2$

Cubic: $F(u) = P_0(1-u)^3 + P_1[3u(1-u)^2] + P_2[3u^2(1-u)] + P_3 u^3$

Degree n: $F(u) = \sum_k P_k B_k^n(u)$ $B_k^n(u)$ are Bernstein-Bezier polynomials

$$B_k^n(u) = \frac{n!}{k!(n-k)!}(1-u)^{n-k} u^k$$

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- Explicit: Bernstein-Bezier polynomial basis
- *4x4 matrix for cubics*
- Properties: Advantages and Disadvantages

Cubic 4x4 Matrix (derive)

$$F(u) = P_0(1-u)^3 + P_1[3u(1-u)^2] + P_2[3u^2(1-u)] + P_3 u^3$$

$$= \begin{pmatrix} u^3 & u^2 & u & 1 \end{pmatrix} \begin{pmatrix} & & & \\ & M = ? & & \\ & & & \\ & & & \end{pmatrix} \begin{pmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \end{pmatrix}$$

Cubic 4x4 Matrix (derive)

$$F(u) = P_0(1-u)^3 + P_1[3u(1-u)^2] + P_2[3u^2(1-u)] + P_3 u^3$$

$$= \begin{pmatrix} u^3 & u^2 & u & 1 \end{pmatrix} \begin{pmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \end{pmatrix}$$

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- *Properties: Advantages and Disadvantages*

Properties (brief discussion)

- Demo: hw2.exe
- Interpolation: End-points, but approximates others
- Single piece, moving one point affects whole curve (no local control as in B-splines later)
- Invariant to translations, rotations, scales etc. That is, translating all control points translates entire curve
- Easily subdivided into parts for drawing (next lecture): Hence, Bezier curves easiest for drawing