

COMS 4160: Problems on Curves

Ravi Ramamoorthi

Questions

1. Consider a quadratic B-spline curve with uniform knot spacing. Consider a segment with control points $(1, 0)$, $(1, 1)$ and $(0, 1)$ in that order. What are the end-points of the curve segment? What is the mid-point of the curve segment?
2. Now repeat the question for a cubic B-spline curve with control points $(-1,-1)$, $(-1,1)$, $(1,1)$, and $(1,-1)$.
3. Using polar forms, and your answers to the evaluations above, derive the general function $f(t)$ for a cubic B-spline, given it's 4 control points. Use this to derive the 4×4 matrix used for cubic B-spline curves.
4. Repeat the question above for a Bezier curve. Can you use this to derive the general Bernstein-Bezier formula for arbitrary degree Bezier curves?
5. We consider the problem of using a Bezier curve to approximate a circle. There exist efficient algorithms to draw Bezier curves, so it is often convenient to reduce other primitives to them. Because of symmetry in a circle, we will consider only the positive quadrant, i.e. with arc endpoints $(1,0)$ and $(0,1)$. What are the control points of a quadratic Bezier curve that best approximates the quarter circle? In particular, the end-points and tangents at those end points of the approximating Bezier curve must match those for the quarter circle. What is the maximum error in this approximation, i.e. the error at the mid-point of the Bezier curve?
6. This is a somewhat advanced question relating Bezier and B-spline curves. In general, since both curves are polynomials, any given actual curve segment can be written as either a Bezier or a B-spline curve of the same degree, but with different control points. First, for the Bezier curve above, give the B-spline control points for an equivalent curve. Second, for the B-spline control points in question 1, what are the corresponding Bezier control points?

Answers

Note that these notes do not for the most part include the deCasteljau diagrams necessary in many cases to do the evaluations. These diagrams will be drawn when these questions are discussed in class. These answers are mainly guides to provide the outlines of the full solutions.

1. Evaluation of quadratic B-spline curve Consider the three control points $P_0 = (1, 0)$, $P_1 = (1, 1)$ and $P_2 = (0, 1)$. Define the corresponding polar form coordinates (using lower case and semicolons to distinguish from upper case and commas for spatial coordinates) as $p_0 = (0; 1)$, $p_1 = (1; 2)$, $p_2 = (2; 3)$. Note that the number of coordinates is proportional to the degree (quadratic) and for B-splines, the numbers depend on the knot sequence, which we assume to be uniform. Also note the difference in convention from

the lecture slides, where we started with -1 , so the curve would lie between 0 and 1. This actually doesn't matter; the curve end-points now correspond to $(1; 1)$ and $(2; 2)$ instead and everything works as before. The midpoint corresponds to $(3/2; 3/2)$. A deCasteljau diagram is perhaps the most convenient method for evaluation and will be shown in class. Here, we detail the main steps. We want to find the intermediate points Q_0 and Q_1 and the final point on the curve.

First, let us consider the left endpoint. The polar form for $q_0 = (1; 1)$ and it corresponds directly to the final value sought. This is clearly given by

$$Q_0 = \frac{P_0 + P_1}{2} = \left(1, \frac{1}{2}\right). \quad (1)$$

By symmetry, the right endpoint is given by the polar form for $q_1 = (2; 2)$ and the value is $Q_1 = (P_1 + P_2)/2 = (\frac{1}{2}, 1)$.

Now, consider the midpoint. We need to use polar form $q_0 = (1; 3/2)$ and $q_1 = (2; 3/2)$ and then take the midpoint. In this case,

$$\begin{aligned} Q_0 &= \frac{P_0 + 3P_1}{4} = \left(1, \frac{3}{4}\right) \\ Q_1 &= \frac{P_2 + 3P_1}{4} = \left(\frac{3}{4}, 1\right) \\ \text{MID} &= \frac{Q_0 + Q_1}{2} = \left(\frac{7}{8}, \frac{7}{8}\right) \end{aligned} \quad (2)$$

2. Evaluation of cubic B-spline curve For the cubic curve, the polar forms are $(0; 1; 2)(1; 2; 3)(2; 3; 4)(3; 4; 5)$ with the desired end points being $(2; 2; 2)(3; 3; 3)$ and the midpoint $(2.5; 2.5; 2.5)$.

First, consider the left endpoint. We will be interested in the 3 intermediate values Q_0, Q_1, Q_2 followed by the 2 at the next step R_0, R_1 . For the left endpoint, we have polar forms

$$q_0 = (1; 2; 2), q_1 = (2; 2; 3), r_0 = (2; 2; 2). \quad (3)$$

We don't care about q_2 or r_1 . Now,

$$Q_0 = \frac{2P_1 + P_0}{3} \quad Q_1 = \frac{2P_1 + P_2}{3}, \quad (4)$$

and the final value

$$R_0 = \frac{Q_0 + Q_1}{2} = \frac{P_0 + 4P_1 + P_2}{6} \quad (5)$$

Similar, for the right end-point, the final value $R_1 = (P_1 + 4P_2 + P_3)/6$. Evaluating these, we obtain that the left and right endpoints are given respectively by $(-2/3, 2/3)$ and $(2/3, 2/3)$.

Now, consider the midpoint. For these, we are interested in polar forms

$$q_0 = (1; 2; 5/2), q_1 = (2; 3; 5/2), q_2 = (3; 4; 5/2). \quad (6)$$

The corresponding points are given by

$$Q_0 = \frac{P_0 + 5P_1}{6} \quad Q_1 = \frac{P_1 + P_2}{2} \quad Q_2 = \frac{5P_2 + P_3}{6}. \quad (7)$$

Evaluating these, we obtain,

$$Q_0 = \left(-1, \frac{2}{3}\right) \quad Q_1 = (0, 1) \quad Q_2 = \left(1, \frac{2}{3}\right). \quad (8)$$

At the next level, the polar forms are $r_0 = (2; 5/2; 5/2)$, $r_1 = (3; 5/2; 5/2)$, and the values are

$$R_0 = \frac{Q_0 + 3Q_1}{4} = \left(-\frac{1}{4}, \frac{11}{12}\right) \quad R_1 = \frac{Q_2 + 3Q_1}{4} = \left(\frac{1}{4}, \frac{11}{12}\right). \quad (9)$$

Finally, the value we seek, which is the midpoint of these, evaluates simply to $(0, 11/12)$.

3. B-spline formula This involves some slightly involved algebra and it is a good exercise to work through it fully. A deCasteljau diagram would certainly help. Here, we outline the main steps.

The four original polar forms p_0, p_1, p_2, p_3 may be written as $(-2; -1; 0)$, $(-1; 0; 1)$, $(0; 1; 2)$, $(1; 2; 3)$. At the next level, we obtain q_0, q_1, q_2 as $(-1; 0; u)$, $(0; 1; u)$, $(1; 2; u)$. The next level will have r_0, r_1 as $(0; u; u)$, $(1; u; u)$ and finally $f(u) = (u; u; u)$ is the function we seek. Now, it is easy to go backwards and simply find all the deCasteljau weights. We obtain the relations:

$$\begin{aligned} Q_0 &= \frac{(1-u)P_0 + (2+u)P_1}{3} & Q_1 &= \frac{(2-u)P_1 + (1+u)P_2}{3} & Q_2 &= \frac{(3-u)P_2 + uP_3}{3} \\ R_0 &= \frac{(1-u)Q_0 + (1+u)Q_1}{2} & R_1 &= \frac{(2-u)Q_1 + uQ_2}{2} \\ f(u) &= (1-u)R_0 + uR_1. \end{aligned} \quad (10)$$

Now, upon simplification of these (which requires considering all possibilities in evaluation, or all paths in the deCasteljau diagram), we obtain

$$f(t) = \frac{1}{6} \left[(1 - 3u + 3u^2 - u^3) P_0 + (3u^3 - 6u^2 + 4) P_1 + (1 + 3u + 3u^2 - 3u^3) P_2 + u^3 P_3 \right], \quad (11)$$

which can be expressed using the classic 4×4 matrix shown in the lecture notes,

$$f(t) = \begin{pmatrix} u^3 & u^2 & u & 1 \end{pmatrix} \frac{1}{6} \begin{pmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 0 & 3 & 0 \\ 1 & 4 & 1 & 0 \end{pmatrix} \begin{pmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \end{pmatrix} \quad (12)$$

4. Bezier formula The bezier formula is in some sense simpler, since the original polar forms are just $(0; 0; 0)$, $(0; 0; 1)$, $(0; 1; 1)$, $(1; 1; 1)$. At the next level, we obtain $(0; 0; u)$, $(0; 1; u)$, $(1; 1; u)$, followed by $(0; u; u)$, $(1; u; u)$ and finally $(u; u; u)$. Similar to the B-spline case, we may write down,

$$\begin{aligned} Q_0 &= (1-u)P_0 + uP_1 & Q_1 &= (1-u)P_1 + uP_2 & Q_2 &= (1-u)P_2 + uP_3 \\ R_0 &= (1-u)Q_0 + uQ_1 & R_1 &= (1-u)Q_1 + uQ_2 \\ f(u) &= (1-u)R_0 + uR_1. \end{aligned} \quad (13)$$

Note the much more regular structure than in the B-spline case, corresponding to the classic deCasteljau construction. In this case, the formula for $f(u)$ doesn't really require that much algebra, and we obtain

$$f(u) = (1-u)^3 P_0 + 3(1-u)^2 u P_1 + 3(1-u) u^2 P_2 + u^3 P_3, \quad (14)$$

whose form immediately suggests that the coefficient in front of P_k is simply the k^{th} term in the binomial expansion of $[(1-u) + u]^n$ where n is the degree of the curve. We may therefore write in general,

$$f_n(u) = \sum_{k=0}^n \frac{n!}{k!(n-k)!} (1-u)^{n-k} u^k P_k, \quad (15)$$

where the first coefficient is simply the relevant binomial factor.

In the particular cubic case, the relevant matrix is obtained by writing equation 14 explicitly,

$$\begin{aligned}
 f(u) &= (1 - 3u + 3u^2 - u^3) P_0 + (3u - 6u^2 + 3u^3) P_1 + (3u^2 - 3u^3) P_2 + u^3 P_3 \\
 &= \begin{pmatrix} u^3 & u^2 & u & 1 \end{pmatrix} \begin{pmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \end{pmatrix}
 \end{aligned} \tag{16}$$

5. Bezier approximation of circle Since the curve must interpolate and be tangent to the end-points, the control points are simply $(1, 0)$, $(1, 1)$, $(0, 1)$. The maximum error in the approximation will occur at the mid-point. Using the deCasteljau diagram or the formula above, we know the mid-point occurs at

$$\frac{P_0 + 2P_1 + P_2}{4} = \left(\frac{3}{4}, \frac{3}{4} \right). \tag{17}$$

The error in this approximation is the difference between this result, and the correct midpoint $(1/\sqrt{2}, 1/\sqrt{2})$. This error is

$$\sqrt{2} \left(\frac{3}{4} - \frac{1}{\sqrt{2}} \right) = \frac{3\sqrt{2}}{4} - 1 \approx 0.06 \tag{18}$$

6. Converting between Bezier and B-spline curves In both cases, the initial control points are $(1,0)$, $(1,1)$ and $(0,1)$. First, for the Bezier curve, the polar form labels are $(0; 0)$, $(0; 1)$ and $(1; 1)$. This can be interpreted as a **non-uniform** B-spline with knot vector $\{0, 0, 1, 1\}$. Note that the knots 0 and 1 are repeated (this is true for any degree; the Bezier curve corresponds to a B-spline with knots at 0 and 1, each repeated as many times as the degree). This repetition of knots ensures the B-spline interpolates its end control points, as required for a Bezier curve (but as is not generally the case with B-splines).

To convert from uniform B-spline to Bezier, we note that the B-spline polar form labels can be taken as $(-1; 0)$, $(0; 1)$ and $(1; 2)$. The way to solve this problem is to use the deCasteljau algorithm to find the control points corresponding to the Bezier polar form labels $(0; 0)$, $(0; 1)$ and $(1; 1)$. The first point $(0; 0)$ is simply the left end-point of the curve and we have already seen in question 1 that it is given by $(1, \frac{1}{2})$. Similarly, the right end-point is $(\frac{1}{2}, 1)$. The second Bezier control point is in this case already a B-spline control point as well, so its value is simply $(1, 1)$. Hence, the corresponding Bezier control points are given by $(\frac{1}{2}, 1)$, $(1, 1)$ and $(1, \frac{1}{2})$. It can easily be verified that the mid-point of this curve is also the mid-point of the B-spline curve found in question 1.