## Computer Graphics (Fall 2004)

COMS 4160, Lecture 3: Transformations 1

http://www.cs.columbia.edu/~cs4160

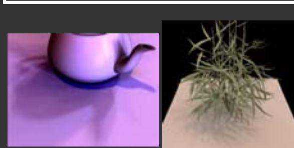
## To Do

- Start (thinking about) assignment 1
  - Much of information you need is in this lecture (slides)
  - Ask TA NOW if compilation problems, visual C++ etc.
  - Not that much coding [solution is approx. 20 lines, but you may need more to implement basic matrix/vector math], but some thinking and debugging likely involved
- Specifics of HW 1
  - Axis-angle rotation derivation and gluLookAt most useful (essential?). These are not covered in text (look at slides).
  - You probably only need final results, but try understanding derivation. Understanding it may make it easier to debug/implement
- Problems in text help understanding material. Usually, we have review sessions per unit, but this one before midterm

## Course Outline

- 3D Graphics Pipeline

**Modeling**
**(Creating 3D Geometry)**
→
**Rendering**
**(Creating, shading images from geometry, lighting, materials)**



## Course Outline

- 3D Graphics Pipeline

**Modeling**
**(Creating 3D Geometry)**
→
**Rendering**
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Unit 1: Transformations
Resizing and placing objects in the world. Creating perspective images.
Weeks 1 and 2 simplestGlut.exe
**Ass 1 due Sep 23 (Demo)**

## Motivation

- Many different coordinate systems in graphics
  - World, model, body, arms, …
- To relate them, we must transform between them
- Also, for modeling objects. I have a teapot, but
  - Want to place it at correct location in the world
  - Want to view it from different angles (HW 1)
  - Want to scale it to make it bigger or smaller

## Motivation

- Many different coordinate systems in graphics
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  - Want to place it at correct location in the world
  - Want to view it from different angles (HW 1)
  - Want to scale it to make it bigger or smaller
- This unit is about the math for doing all these things
  - Represent transformations using matrices and matrix-vector multiplications.
- **Demo: HW 1, applet** transformation_game.jar

## General Idea

- Object in model coordinates
- Transform into world coordinates
- Represent points on object as vectors
- Multiply by matrices
- Demos with applet

- Chapter 5 in text. We cover most of it essentially as in the book. Worthwhile (but not essential) to read whole chapter

## Outline

- *2D transformations: rotation, scale, shear*
- Composing transforms
- 3D rotations
- Translation: Homogeneous Coordinates (next time)
- Transforming Normals (next time)

## (Nonuniform) Scale

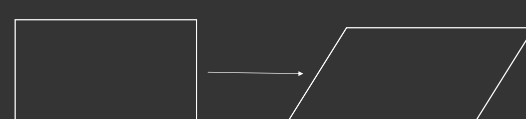
$$Scale(s_x, s_y) = \begin{pmatrix} s_x & 0 \\ 0 & s_y \end{pmatrix} \qquad S^{-1} = \begin{pmatrix} s_x^{-1} & 0 \\ 0 & s_y^{-1} \end{pmatrix}$$

$$\begin{pmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & s_z \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} s_x x \\ s_y y \\ s_z z \end{pmatrix}$$

transformation_game.jar

## Shear

$$Shear = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \qquad S^{-1} = \begin{pmatrix} 1 & -a \\ 0 & 1 \end{pmatrix}$$



## Rotations

2D simple, 3D complicated. [Derivation? Examples?]

2D? $$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

- Linear R(X+Y)=R(X)+R(Y)
- Commutative

transformation_game.jar

## Outline

- 2D transformations: rotation, scale, shear
- *Composing transforms*
- 3D rotations
- Translation: Homogeneous Coordinates
- Transforming Normals

## Composing Transforms

- Often want to combine transforms
- E.g. first scale by 2, then rotate by 45 degrees
- Advantage of matrix formulation: All still a matrix
- Not commutative!! Order matters

## E.g. Composing rotations, scales

$$x_3 = Rx_2 \qquad x_2 = Sx_1$$

$$x_3 = R(Sx_1) = (RS)x_1$$

$$x_3 \neq SRx_1$$

transformation_game.jar

## Inverting Composite Transforms

- Say I want to invert a combination of 3 transforms
- Option 1: Find composite matrix, invert
- Option 2: Invert each transform ***and swap order***
- Obvious from properties of matrices

$$M = M_1 M_2 M_3$$

$$M^{-1} = M_3^{-1} M_2^{-1} M_1^{-1}$$

$$M^{-1}M = M_3^{-1}(M_2^{-1}(M_1^{-1}M_1)M_2)M_3$$

transformation_game.jar

## Outline

- 2D transformations: rotation, scale, shear
- Composing transforms
- *3D rotations*
- Translation: Homogeneous Coordinates
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## Rotations

Review of 2D case

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

- Orthogonal?, $R^T R = I$

## Rotations in 3D

- Rotations about coordinate axes simple

$$R_z = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad R_x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$$

$$R_y = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}$$

- Always linear, orthogonal
  - Rows/cols orthonormal

$R^T R = I$

R(X+Y)=R(X)+R(Y)

## Geometric Interpretation 3D Rotations

- Rows of matrix are 3 unit vectors of new coord frame
- Can construct rotation matrix from 3 orthonormal vectors

$$R_{uvw} = \begin{pmatrix} x_u & y_u & z_u \\ x_v & y_v & z_v \\ x_w & y_w & z_w \end{pmatrix} \qquad u = x_u X + y_u Y + z_u Z$$

$$Rp = \begin{pmatrix} x_u & y_u & z_u \\ x_v & y_v & z_v \\ x_w & y_w & z_w \end{pmatrix} \begin{pmatrix} x_p \\ y_p \\ z_p \end{pmatrix} = ? \qquad \begin{pmatrix} u \bullet p \\ v \bullet p \\ w \bullet p \end{pmatrix}$$

## Geometric Interpretation 3D Rotations

$$Rp = \begin{pmatrix} x_u & y_u & z_u \\ x_v & y_v & z_v \\ x_w & y_w & z_w \end{pmatrix} \begin{pmatrix} x_p \\ y_p \\ z_p \end{pmatrix} = \begin{pmatrix} u \bullet p \\ v \bullet p \\ w \bullet p \end{pmatrix}$$

- Rows of matrix are 3 unit vectors of new coord frame
- Can construct rotation matrix from 3 orthonormal vectors
- Effectively, projections of point into new coord frame
- New coord frame uvw taken to cartesian components xyz
- Inverse or transpose takes xyz cartesian to uvw

## Non-Commutativity

- Not Commutative (unlike in 2D)!!
- Rotate by x, then y is not same as y then x
- Order of applying rotations does matter
- Follows from matrix multiplication not commutative
  - R1 * R2 is not the same as R2 * R1
- Demo: HW1, order of right or up will matter
  - simplestGlut.exe

## Arbitrary rotation formula

- Rotate by an angle θ about arbitrary axis **a**
  - Not in book. (see bottom page 98, but not very complete)
  - Homework 1: must rotate eye, up direction
  - Somewhat mathematical derivation, but useful formula
- Problem setup: Rotate vector **b** by θ about **a**
- Helpful to relate **b** to X, **a** to Z, verify does right thing
- For HW1, you probably just need final formula

simplestGlut.exe

## Warnings and Caveats

- The derivation is quite involved mathematically
  - Don't focus on math details (but they are here for those who are particularly interested).
  - Instead, see if you can understand the very basic steps
  - This section is more for you, if you are interested. This material was covered quickly in class and won't be tested
- Common operation
  - In practice, such as in HW 1, you do often need to rotate by an arbitrary vector. So, the final formula is good to know
  - Though in practice, you'll likely use a canned routine like setRot or glRotate that implements it directly

## Axis-Angle formula

- Step 1: **b** has components parallel to **a**, perpendicular
  - Parallel component unchanged (rotating about an axis leaves that axis unchanged after rotation, e.g. rot about z)
- Step 2: Define **c** orthogonal to both **a** and **b**
  - Analogous to defining Y axis
  - Use cross products and matrix formula for that
- Step 3: With respect to the perpendicular comp of **b**
  - Cos θ of it remains unchanged
  - Sin θ of it projects onto vector **c**
  - Verify this is correct for rotating X about Z
  - Verify this is correct for θ as 0, 90 degrees

## Axis-Angle formula 1(derive on board)

- Step 1: **b** has components parallel to **a**, perpendicular
  - Parallel component unchanged (rotating about an axis leaves that axis unchanged after rotation, e.g. rot about z)

$$b \to a = (a \bullet b)a = a(a \bullet b) = aa^T b = (aa^T)b$$
$$b \setminus a \quad = b - (aa^T)b = (I_{3\times 3} - aa^T)b$$

## Axis-Angle formula 2(from last lecture)

- Step 2: Define **c** orthogonal to both **a** and **b**
  - Analogous to defining Y axis
  - Use cross products and matrix formula for that

$$c = a \times b = A^* b \qquad A^* = \begin{pmatrix} 0 & -z_a & y_a \\ z_a & 0 & -x_a \\ -y_a & x_a & 0 \end{pmatrix}$$

Dual matrix of vector a

## Axis-Angle formula 3

- Step 3: With respect to the perpendicular comp of **b**
  - Cos θ of it remains unchanged
  - Sin θ of it projects onto vector **c**

$$(b \setminus a)_{ROT} = (b \setminus a)\cos\theta + c\sin\theta$$

$$b \setminus a = (I_{3\times 3} - aa^T)b$$
$$c \quad = A^* b$$

$$(b \setminus a)_{ROT} = (I_{3\times 3}\cos\theta - aa^T\cos\theta)b + (A^*\sin\theta)b$$

## Axis-Angle: Putting it together (derive)

$$(b \setminus a)_{ROT} = (I_{3\times 3}\cos\theta - aa^T\cos\theta)b + (A^*\sin\theta)b$$
$$(b \to a)_{ROT} = (aa^T)b$$

$$R(a,\theta) = I_{3\times 3}\cos\theta + aa^T(1-\cos\theta) + A^*\sin\theta$$

$$R(a,\theta) = \cos\theta \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + (1-\cos\theta)\begin{pmatrix} x^2 & xy & xz \\ xy & y^2 & yz \\ xz & yz & z^2 \end{pmatrix} + \sin\theta \begin{pmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{pmatrix}$$