COMS 6998-006 Network Theory **Columbia University, Spring 2008**

Lecture 7, Part 2 - self similarity: Feb 14, 2008

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1 Self-similarity and Fractals

1.1 Self-similarity

A self-similar object is exactly or approximately similar to a part of itself, e.g., the whole has the same shape as one or more of the parts.

1.2 Fractals

Self-similarity is a typical property of fractals. Here is an example of fractal and how it evolves.
We start with a square, cut it into nine tiles, and shadow the one in the middle. In every iteration, cut each remaining (i.e. unshadowed) tile into nine tiles and shadow the center. In this way each tile is a smaller version of the initial square.

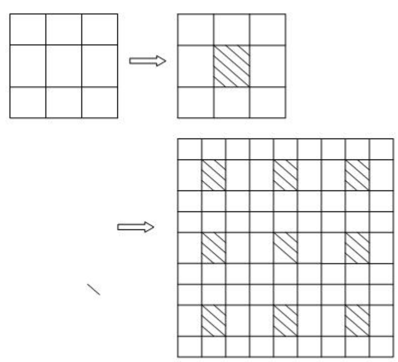

Figure 1: Evolution of Fractals

Figure 1 shows how a fractal evolves. The remaining area = 0, when n$\rightarrow\infty$.

1.3 Another example - Sierpinski Gasket

Sierpinski Gasket (Figure 2) is another example of the fractals.Originally constructed as a curve, this is one of the basic examples of self-similar sets, i.e. it is a mathematically generated pattern that can be reproducible at any magnification or reduction.

An algorithm for obtaining arbitrarily close approximations to the Sierpinski triangle is as follows:

1) Start with any triangle in a plane (any closed, bounded region in the plane will actually work). The canonical Sierpiĺ¡ski triangle uses an equilateral triangle with a base parallel to the horizontal axis (first image).

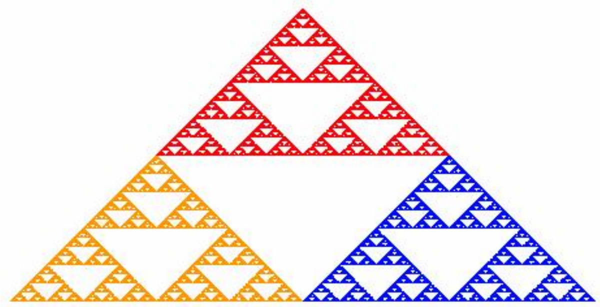

Figure 2: Sierpinski Gasket

2) Shrink the triangle to 1/2 height and 1/2 width, make two copies, and position the three shrunken triangles so that each triangle touches the two other triangles at a corner (image 2). Note the emergence of the central hole - because the three shrunken triangles can between them cover only 3/4 of the area of the original. (Holes are an important feature of Sierpiĺ¡ski's Triangle.
3) Repeat step 2 with each of the smaller triangles (image 3 and so on).

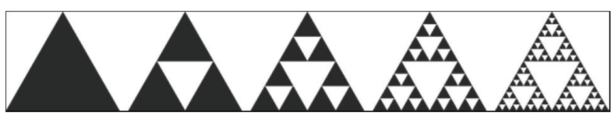

Figure 3: Evolution of Sierpinski Gasket

The evolution process is illustrated in Figure 3.

2 Properties of Fractals

2.1 Calculate the cut area by Cantor set

In mathematics, the Cantor set is a set of points lying on a single line segment that has a number of remarkable and deep properties.
Calculating the cut area: (Figure 4)

Figure 4: Counting with Cantor set

$$Cutarea = 1/3 + 2/9 + 4/27 + ... = \sum_{i=1}^{\infty} x^{(i-1)}/3^{i} = \frac{1}{3} * \frac{1}{1 - 2/3} = 1 \quad (1)$$

which means we will cut all the areas if iteratively repeat the operations.

2.2 Determine the dimension of the Gasket - by box counting

Definition: The Minkowski-Bouligand dimension or Minkowski dimension is a way of determining the fractal dimension of a set S in a Euclidean space. This dimension is also, less accurately, sometimes known as the packing dimension or the box-counting dimension.

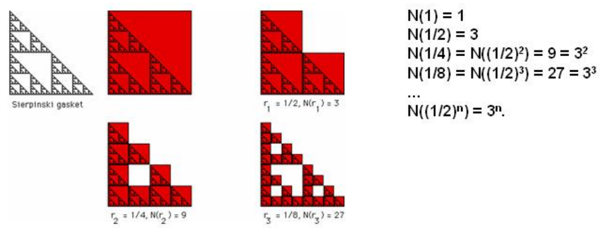

Figure 5: Box Counting(With permission from Professor Daniel Spielman at Yale University)

Determine the dimension of the Gasket according to the formula below: (Figure 5)

$$dim(S) := \lim_{\epsilon \to 0} \frac{\log N(\epsilon)}{\log(1/\epsilon)}$$

N(ϵ) is the number of boxes of side length ϵ required to cover the set.

3 Readings

Here is a list of useful wikipedia pages.

1. en.wikipedia.org/wiki/Fractal.
2. en.wikipedia.org/wiki/Gasket
3. en.wikipedia.org/wiki/Sierpinski_triangle
4. en.wikipedia.org/wiki/Box-counting_dimension