COMS 6998-006 Network Theory	**Columbia University, Spring 2008**

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1 Small World Networks

In the previous lectures we have mentioned several instances of small world networks[7]: social networks (network based on the co-appearances in movies with Kevin Bacon[5], network based on the Erdös Number[3], etc.), protein networks, etc.. This lecture is dedicated to the formal study of small world networks and how one can generate such networks. As we shall see the idea of small world networks initially emerged through real-world experiments[6]. We present two models describing the generation of small world networks, that is networks exhibiting the following two characteristics: (1) a low diameter and (2) a high clustering coefficient.

1.1 Small World Experiments

Milgram's Experiment The first experiments of this kind dates back to the 60s with the research conducted by Stanley Milgram. Milgram analyzed the average path lenghth for social networks of people in the United States. In his most famous experiment [8], Milgram selected 296 random individuals from two US cities (Omaha, Nebraska and Wichita, Kansas), asking them to forward an initial letter to a target contact person based in Boston. Participants were asked to either forward the letter to the target if they actually knew the person, or to forward it to a person of their acquaintance that was the most likely to know the target contact person. Results: only 20 percent of the packages sent reached their target, yielding an average chain length of 6.5 (although this number does not take into account the remaining 80 percent of the packages).

More Recently The experiment was later reproduced by Dodds [4] - using email as a medium - at a more global scale (18 targets, 13 countries, 60K participants) which resulted in 384 messages reaching their target, yielding an average path length of 4.

1.2 The Watts-Strogatz model

We now look at the modelizing of small world networks. Small world networks can be seen as an intermediary step between regular lattices (characterized by high clustering coefficients and high diameters) and random graphs (characterized by low clustering coefficients but small diameters). Watts & Strogatz define small-world networks as being graphs that exhibit high clustering coefficient (like regular lattices) but low diameter (like random graphs). Based on this definition they propose a method to grow a random graph while keeping its diameter small.

The intuition behind the Watts-Strogatz model is to start from a regular lattice and to progressively rewire it, one edge after another, until we obtain a random graph (c.f. Figure 1). By rewiring we mean the action of, given an edge in the graph, disconnect it from its end point and rewire it to a random node in the graph. During this process, the farther the node being reconnected to, the more the graph diameter is reduced (note that at any step, the best reduction ratio that can be achieved is $\frac{1}{2}$). After only a few steps the diameter of the graph has been reduced dramatically while its clustering coefficient remains large.

To analyze the rewiring process we characterize a network using two parameters:

- coordination number z: the number of neighbors for each node.

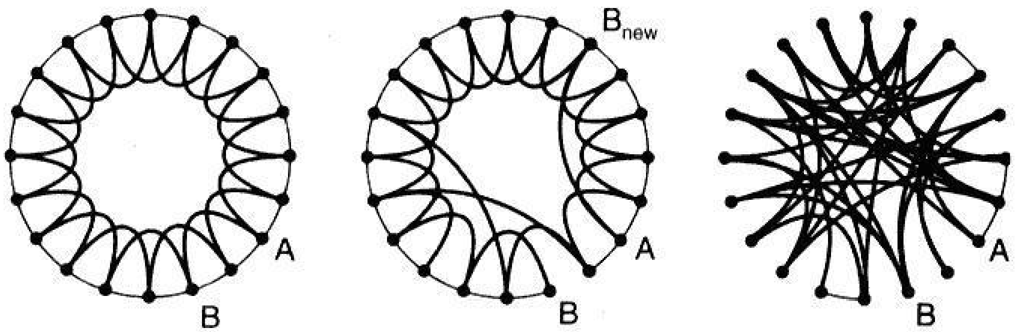

Figure 1: Rewiring of a regular lattice. From left to right: regular lattice; graph after only a few edge rewiring; random graph.[9]

- shortcut probability p: for an existing edge, the probability to draw a shortcut between two random nodes.

The total (expected) number of shortcuts in the network can then be computed as $m * p = \frac{n*z*p}{2}$.

1.2.1 Impact of the shortcuts probability on the diameter

Amaral and Barthélémy[2] report that for an initial lattice with $N = 1000$ vertices and with a coordination number $z = 10$, they obtain a network with diameter $d = 7.6$ when $p = 0.016$ while the diameter would drop to $d = 3.6$ when $p = 0.25$.

1.2.2 Clustering coefficient analysis

The clustering coefficient mirrors the structure of the underlying lattice. According to [1] the clustering coefficient C can in fact be modeled as:

$$C = \frac{3(z-1)}{2(2z-1)} * (1-p)^3 \tag{1}$$

reaching the $C = \frac{3}{4}$ limit when $p \to 0$ (c.f. Figure 2).

1.3 Kleinberg model

As we saw, the Watts-Strogatz model uses a constant shortcuts probability, regardless of whether the node being rewired to is close to or remote from the source. Kleinberg proposes a refinement where the geographical distance between two nodes affects the probability of an edge being rewired between them (e.g. $p \sim \frac{1}{d^2}$).

1.4 Readings

1. All of [9]
2. Pages 1-8 of [1]

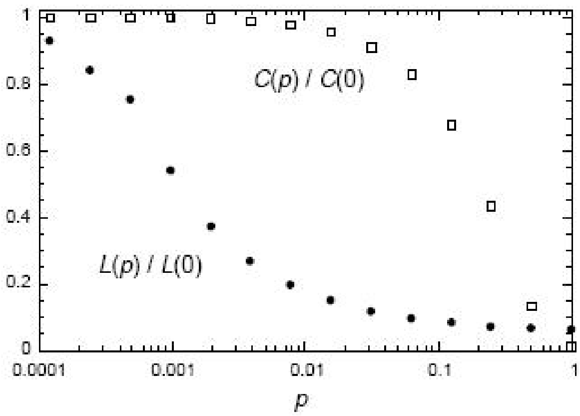

Figure 2: Evolution of the diameter and clustering coefficients as a function of the shortcuts probability, from regular lattice (leftmost), to small-world networks (center), to random networks (rightmost)

References

[1] Alain Barrat and M. Weigt. "On the properties of small-world network models". In: *The European Physical Journal B* 13 (2000). Pp. 547–560.

[2] Marc Barthélémy and Luis A. Nunes Amaral. "Small-World Networks: Evidence for a Crossover Picture". In: *Phys. Rev. Lett.* 82 (1999). Pp. 3180–3183.

[3] *Erdös number*. URL: `http://en.wikipedia.org/wiki/Erdos_number`.

[4] Roby Muhamad Peter Sheridan Dodds and Duncan J. Watts. "An Experimental Study of Search in Global Social Networks". In: *Science* 301.5634 (Aug. 2003). Pp. 827–629.

[5] *Six Degrees of Kevin Bacon*. URL: `http://en.wikipedia.org/wiki/Six_degrees_of_Kevin_Bacon`.

[6] *Small world experiment*. URL: `http://en.wikipedia.org/wiki/Small_world_experiment`.

[7] *Small-world network*. URL: `http://en.wikipedia.org/wiki/Small-world_network`.

[8] Jeffrey Travers and Stanley Milgram. "An Experimental Study of the Small World Problem". In: *Sociometry* 32.4 (1969). Pp. 425–443.

[9] Duncan J. Watts and Steven H. Strogatz. "Collective dynamics of 'small-world' networks". In: *Nature* 393 (June 1998). Pp. 440–442.