COMS 6998-006 Network Theory **Columbia University, Spring 2008**

Lecture 22 - The Ising Model and Percolation on Graphs. April 11, 2008

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1 The Ising Model and Percolation

Percolation is the simplest natural phenomenom that models phase transitions.
The basic setup is analogous to the following: say a porous stone is dipped partially in water. Is there a path for water to flow up to the top of stone? There are two kinds of percolation models: bond percolation and site percolation.

1.1 Bond Percolation

Consider a lattice consisting of n nodes. Edges represent open paths, and an edge exists between two nodes with probability p. Figure 1 is an example of the bond percolation setup. Note that in this example, there exists a path from the bottom to the top (highlighted in blue).

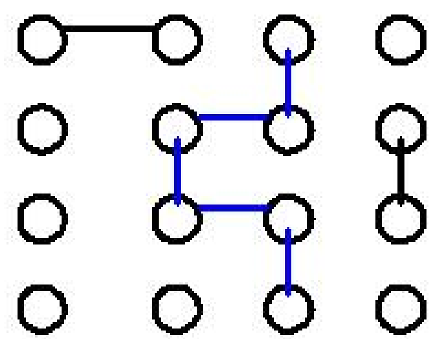

Figure 1: An example of bond percolation. The lattice is made of nodes (the circles), and edges exist between the nodes with probablity p. An edge symbolizes an open path. There is a path from the bottom to the top (blue), and thus percolation exists.

1.2 Site Percolation

In the site percolation model, a lattice consists of n positions. Each position is either filled with a node, or not, and nodes exist with probability p. In Figure 2, a space is occupied if the node is coloured, and is empty if not. Again, in the figure, there exists a path from the bottom to the top.

1.3 Phase Transition

Consider the effect of changing p on whether percolation occurs:

- $p = 0$: In this case, the probability of percolation =0, because no edges are added (or no spaces are filled).

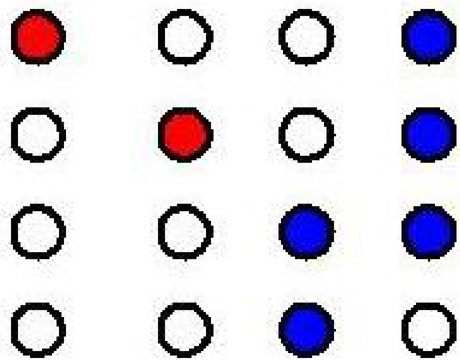

Figure 2: An example of site percolation. In this case, the lattice consists of empty spaces (unfilled circles). These spaces are filled with a node with probability p (filled circles). In this case, there exists a path from the bottom to the top (if you follow the blue circles).

- $p = 1$: In this case, the probability of percolation =1, because all edges are added (or, all spaces are filled).

This suggests that the probability of percolation increases with p. In fact, there is an important value of p (the threshhold value) above which the probability of percolation grows very quickly until it reaches 1.
This abrupt transition of the behaviour of a system at some critical value is called a phase transition ([3].
Figure 3 is a plot of p versus probability of percolation.

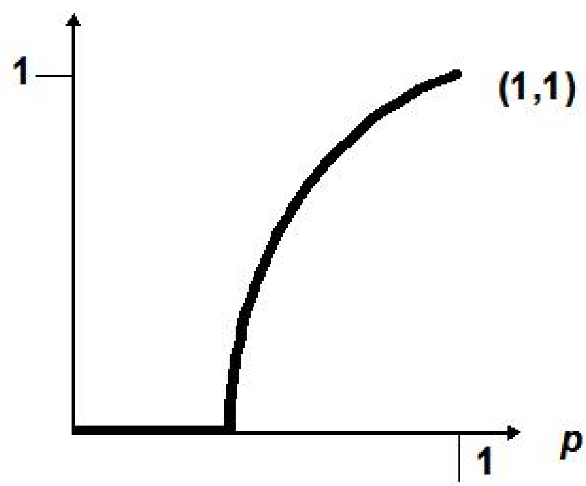

Figure 3: The phase transition plot for modeling percolation. For values of p lower than some threshold value (x axis), the probability of percolation (y axis) is 0. Over the threshold though, the probability of percolation rises very steeply to 1.

1.4 Curie Point

The Curie point is a property of ferromagnetic materials ([6]).
The property asserts that if a ferromagnetic object is heated to a temperature higher than its Curie point, it will lose its ferromagnetic property. The Ising model, explained below, is used to describe the phase transition model of ferromagnetic materials subjected to heat.

1.5 The Ising Model

Consider a two dimensional array, where each cell holds a value of +1 or -1. A specific assignment of +/- 1s to each cell is called a configuration. The energy associated with a configuration is given by,
$E = -sum_{ij}(S_{i,j}S_{i,j+1} + S_{i,j}S_{i+1,j})$
where $S_{i,j}$ is the value (+/- 1) at position (i, j). From the equation, it is clear that a lower energy configuration is one where each node and its immediate neighbours have the same spin, or the same value.
If the +/-1 are thought of as spins, this model can be used to describe the demagnetizing of ferromagnetic materials when subject to heat. The model is of phase transition, as described in Section 1.4 (see Figure 4.

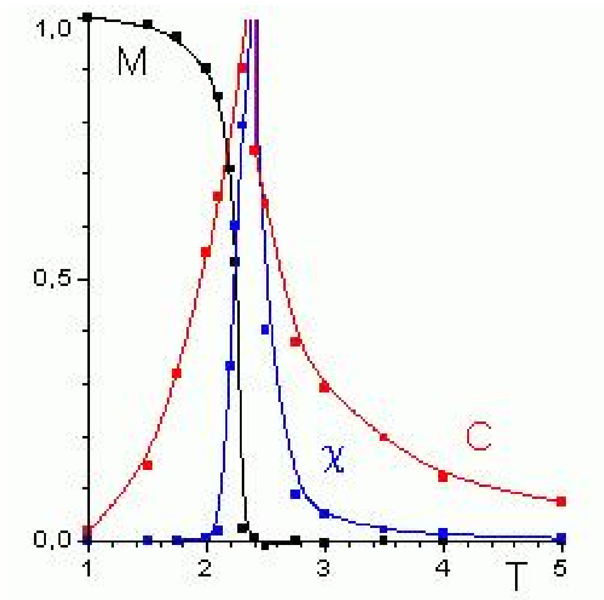

Figure 4: A plot of magnetism (M), susceptibility (χ), and specific heat (C) as a function of temperature ([1]). The magnetism plot follows a phase transition model. At a critical temperature, the magnetism drops pretty sharply to zero.

1.6 Demos

There are a number of demos online that help make the concept of percolation clearer. [2] allows you to set a temperature and then run a simulation that visualizes the spins on a lattice as well as the magnetism of the lattice. [5] also plots the values of E and M over time.

1.7 Readings

[4]

References

[1] "Ferromagnetic phase transition". In: (). URL: `http://www.ibiblio.org/e-notes/Perc/trans.htm`.

[2] "Ising Model Main Page". In: (). URL: `http://webphysics.davidson.edu/applets/ising/default.html`.

[3] Harry Kesten. "What is Percolation?". In: *Notices of the AMS* 53.5 (2006).

[4] Cristopher Moore and M. E. J. Newman. "Epidemics and percolation in small-world networks". In: *Physical Review E* 61 (2000). P. 5678. URL: http://www.citebase.org/abstract?id=oai:arXiv.org:cond-mat/9911492.

[5] "Simulations for Statistical and Thermal Physics". In: (). URL: http://stp.clarku.edu/simulations/ising/ising2d.html.

[6] Wikipedia. "Curie Point – Wikipedia, the free encyclopedia". In: (2004). URL: http://en.wikipedia.org/wiki/Curie_point.