Exercise 1 (10 points): Consider projective measurements in the computational basis.

a.) A single qubit matrix $U$ has the eigenvalues $\pm 1$ and the orthonormal eigenvectors $|\psi_1\rangle, |\psi_2\rangle$, i.e. we can write it as,

$$U = |\psi_1\rangle \langle \psi_1| - |\psi_2\rangle \langle \psi_2|.$$ 

Consider the following circuit (the M in the circle is a measurement):

\[\begin{array}{c}
|0\rangle \quad H \quad H \quad M \rightarrow |m_{out}\rangle \\
|\psi_{in}\rangle \quad U \quad \quad \quad \quad \rightarrow |\psi_{out}\rangle
\end{array}\]

Find $|m_{out}\rangle$ and $|\psi_{out}\rangle$, and explain how they are related to $|\psi_1\rangle$ and $|\psi_2\rangle$.

b.) Prove that measurement commutes with controls. That is, for an arbitrary single qubit unitary $U$ we have

\[\begin{array}{c}
M \quad = \\
U
\end{array}\]

Exercise 2 (10 points):

Let $f$ be a function $f : \{0, 1, \ldots, 2^n - 1\} \rightarrow \{0, 1, \ldots, 2^m - 1\}$ and

$$U_f |j\rangle |k\rangle = |j\rangle |k \oplus f(j)\rangle,$$

with $|j\rangle |k\rangle \in \mathbb{C}^{2^n} \otimes \mathbb{C}^{2^m}$ and $\oplus$ denoting addition modulo $2^m$.

For an arbitrary integer $\ell \geq 0$ find the matrix representation of

$$U_f^{\ell} = U_f U_f \cdots U_f.$$
Exercise 3 (10 points):

a.) Write a program to simulate the Deutsch-Josza quantum algorithm. Here "simulate" means that your algorithm actually does the same steps as in the quantum algorithm and not only returns a precomputed answer.

b.) Suppose that a Boolean function \( f: \{0, 1, \ldots, 2^n - 1\} \rightarrow \{0, 1\} \), with \( n \geq 2 \), is constant, i.e., \( f(j) = f(0) \), or \( \frac{1}{4} \)-balanced, i.e., \( |\{j \mid f(j) = 1\}| = \frac{3}{4}2^n \).

Modify the Deutsch-Josza algorithm to check that \( f \) is constant with probability \( p > \frac{1}{2} \).

c.) Write a program to simulate the quantum algorithm of b.).