W4281 - Introduction to Quantum Computing

Homework 1

due date: Thursday 6/2/2005

Note: for this homework we use the 6 single qubit gates $X$, $Y$, $Z$ (Pauli matrices), $H$ (Hadamard gate), $S$ (phase gate), and $T$ ($\pi/8$ gate) as defined in class or in Nielsen and Chuang, p. 174.

Exercise 1 (10 points):
In this problem you compute the exponential $e^M$ of a matrix $M$. The definition of the exponential of a matrix can be found in Nielsen and Chuang, p. 75.

Let $X$, $Y$, $Z$ be the Pauli matrices and let $\alpha$ be a complex number. Compute

$e^{\alpha X}$, $e^{\alpha Y}$, $e^{\alpha Z}$.

Exercise 2 (10 points):
Show that the Hadamard gate $H$ can be written as

$$H = \frac{1}{\sqrt{2}} (|0\rangle + |1\rangle) \langle 0| + \frac{1}{\sqrt{2}} (|0\rangle - |1\rangle) \langle 1|$$

and that the $n$-fold tensor product of $H$ with itself is

$$H^\otimes n = \frac{1}{2^n/2} \sum_{j,k=0}^{2^n-1} (-1)^{j \cdot k} |j\rangle \langle k|$$

where $j \cdot k$ denotes

$$j_0 k_0 + j_1 k_1 + \cdots + j_{n-1} k_{n-1}$$

if the binary expansions of $j$ and $k$ are

$$j = (j_{n-1} j_{n-2} \cdots j_0)_2 = \sum_{l=0}^{n-1} 2^l j_l$$

$$k = (k_{n-1} k_{n-2} \cdots k_0)_2 = \sum_{l=0}^{n-1} 2^l k_l.$$
Exercise 3 (10 points):
Write a program to solve the following problem:

Input: Three integers $j, k, m$, with $m, j \in \{0, 1, \ldots, 2^k - 1\}$.

Output: The probability to measure the outcome $j$ of the $k$-qubit state

$$|\psi\rangle = U_1 \otimes U_2 \otimes \ldots \otimes U_k |m\rangle,$$

where the $U_l, l \in \mathbb{N}$, are the following matrices:

- $U_1 = U_7 = \ldots = X$
- $U_2 = U_8 = \ldots = Y$
- $U_3 = U_9 = \ldots = Z$
- $U_4 = U_{10} = \ldots = H$
- $U_5 = U_{11} = \ldots = S$
- $U_6 = U_{12} = \ldots = T$.

Your algorithm should be linear in $k$.

Hint: You only have to compute the probability for the $j$ given as input. This probability is given by

$$|\langle j | U_1 \otimes U_2 \otimes \ldots \otimes U_k |m\rangle|^2.$$

Example: For $j = 2, k = 2, m = 1$ this probability is

$$|\langle 2 | X \otimes Y |1\rangle|^2 = |\langle 10_2 | X \otimes Y |01_2\rangle|^2 = | -i \langle 10_2 | 10_2\rangle|^2 = |-i|^2 = 1.$$