Data Structures and Algorithms

Session 20. April 8, 2009

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Announcements

- Homework 5 is out:
  - You can use adjacency lists if you prefer

Review

- Extensions of Dijkstra's Algorithm
  - Critical Path Analysis
  - All Pairs Shortest Path (Floyd-Warshall)
- Maximum Flow
  - Floyd-Fulkerson Algorithm

Today's Plan

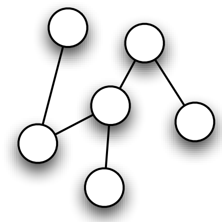
- Minimum Spanning Tree
  - Prim's Algorithm
  - Kruskal's Algorithm
- Depth first search
  - Euler Paths

Minimum Spanning Tree Problem definition

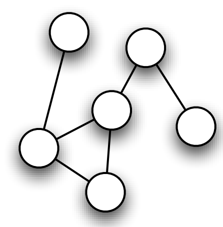
- Given connected graph **G**, find the connected, acyclic subgraph **T** with minimum edge weight
  - A tree that includes every node is called a **spanning tree**
- The method to find the MST is another example of a greedy algorithm

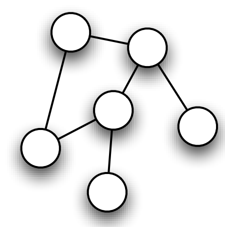
Motivation for Greed

- Consider any spanning tree

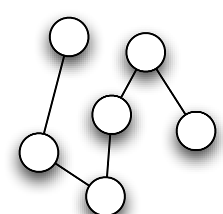


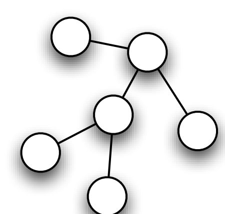
- Adding another edge to the tree creates exactly one cycle





- Removing an edge from that cycle restores the tree structure





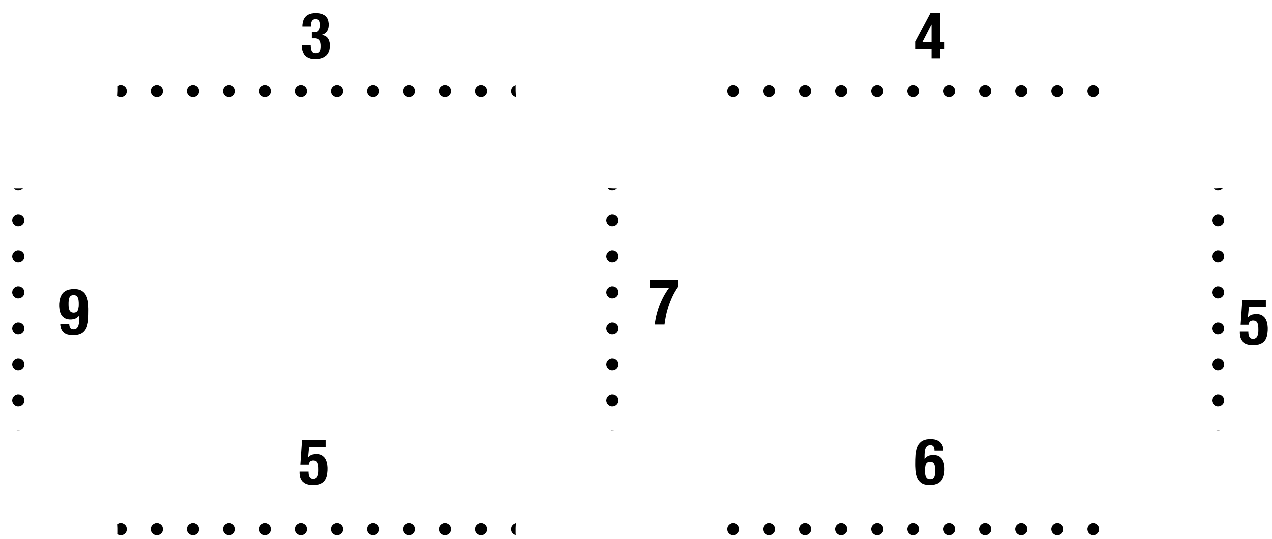
Prim’s Algorithm

- Grow the tree like Dijkstra’s Algorithm
- Dijkstra’s: grow the set of vertices to which we know the shortest path
- Prim’s: grow the set of vertices we have added to the minimum tree
- Store shortest edge **D**[ ] from each node to tree

Prim’s Algorithm

- Start with a single node tree, set distance of adjacent nodes to edge weights, infinite elsewhere
- Repeat until all nodes are in tree:
  - Add the node **v** with shortest known distance
  - Update distances of adjacent nodes **w**:
    D[w] = min(**D**[w], weight(**v**,**w**))

Prim’s Example



Implementation Details

- Store "previous node" like Dijkstra's Algorithm; backtrack to construct tree after completion
- Of course, use a priority queue to keep track of edge weights. Either
    - keep track of nodes inside heap & decreaseKey
    - or just add a new copy of the node when key decreases, and call deleteMin until you see a node not in the tree

Prim's Algorithm Justification

- At any point, we can consider the set of nodes in the tree **T** and the set outside the tree **Q**
- Whatever the MST structure of the nodes in **Q**, at least one edge must connect the MSTs of **T** and **Q**
- The greedy edge is just as good structurally as any other edge, and has minimum weight

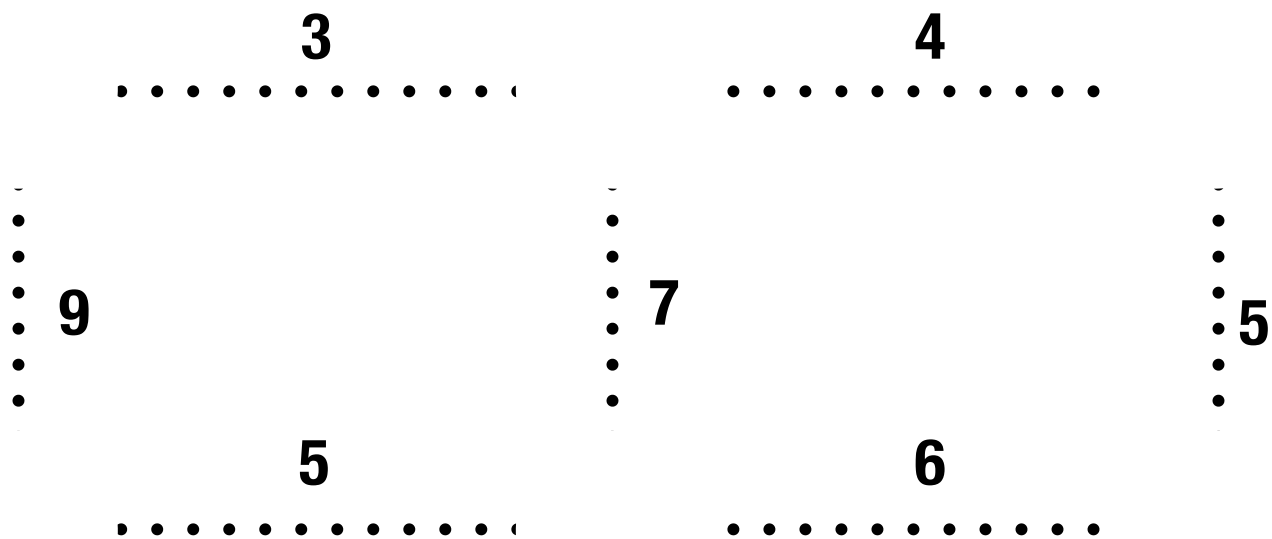
Prim's Running Time

- Each stage requires one deleteMin $O(\log |V|)$, and there are exactly $|V|$ stages
- We update keys for each edge, updating the key costs $O(\log |V|)$ (either an insert or a decreaseKey)
- Total time: $O(|V| \log |V| + |E| \log |V|) = O(|E| \log |V|)$

Kruskal's Algorithm

- Somewhat simpler conceptually, but more challenging to implement
- Algorithm: repeatedly add the shortest edge that does not cause a cycle until no such edges exist
- Each added edge performs a union on two trees; perform unions until there is only one tree
- Need special ADT for unions (Disjoint Set... we'll cover it later)

Kruskal's Example



Kruskal’s Justification

- At each stage, the greedy edge **e** connects two nodes **v** and **w**
- Eventually those two nodes must be connected;
  - we must add an edge to connect trees including **v** and **w**
- We can always use **e** to connect **v** and **w**, which must have less weight since it's the greedy choice

Kruskal's Running Time

- First, buildHeap costs O(|E|)
- Each edge, need to check if it creates a cycle (costs O(log V))
- In the worst case, we have to call |E| deleteMins
- Total running time O(|E| log |E|); but $|E| \leq |V|^2$

$$O(|E| \log |V|^2) = O(2|E| \log |V|) = O(|E| \log |V|)$$

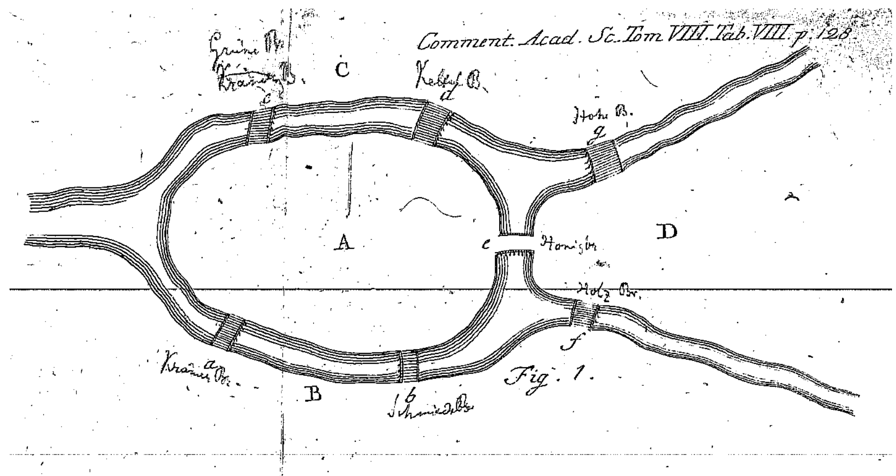
MST Wrapup

- Connect all nodes in graph using minimum weight tree
- Two greedy algorithms:
  - Prim's: similar to Dijkstra's. Easier to code
  - Kruskal's: easy on paper

Depth First Search

- Level-order <–> Breadth-first Search
  Preorder <–> Depth-first Search
- Visit vertex **v**, then recursively visit **v**'s neighbors
- To avoid visiting nodes more than once in a cyclic graph, mark visited nodes,
- and only recurse on unmarked nodes

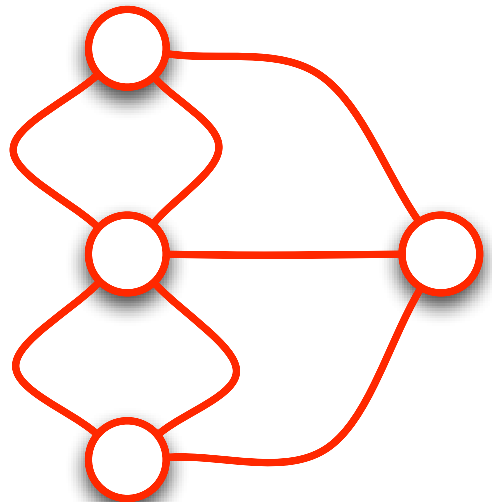
The Seven Bridges of Königsberg



http://math.dartmouth.edu/~euler/docs/originals/E053.pdf

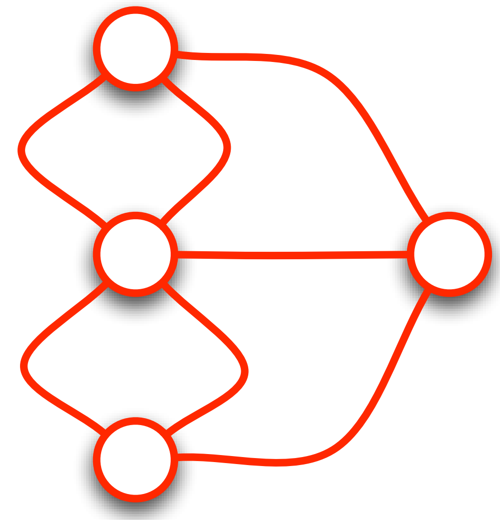
- Königsburg Bridge Problem: can one walk across the seven bridges and never cross the same bridge twice?

Euler Paths and Circuits



- Euler path – a (possibly cyclic) path that crosses each edge exactly once
- Euler circuit - an Euler path that starts and ends on the same node

Euler's Proof

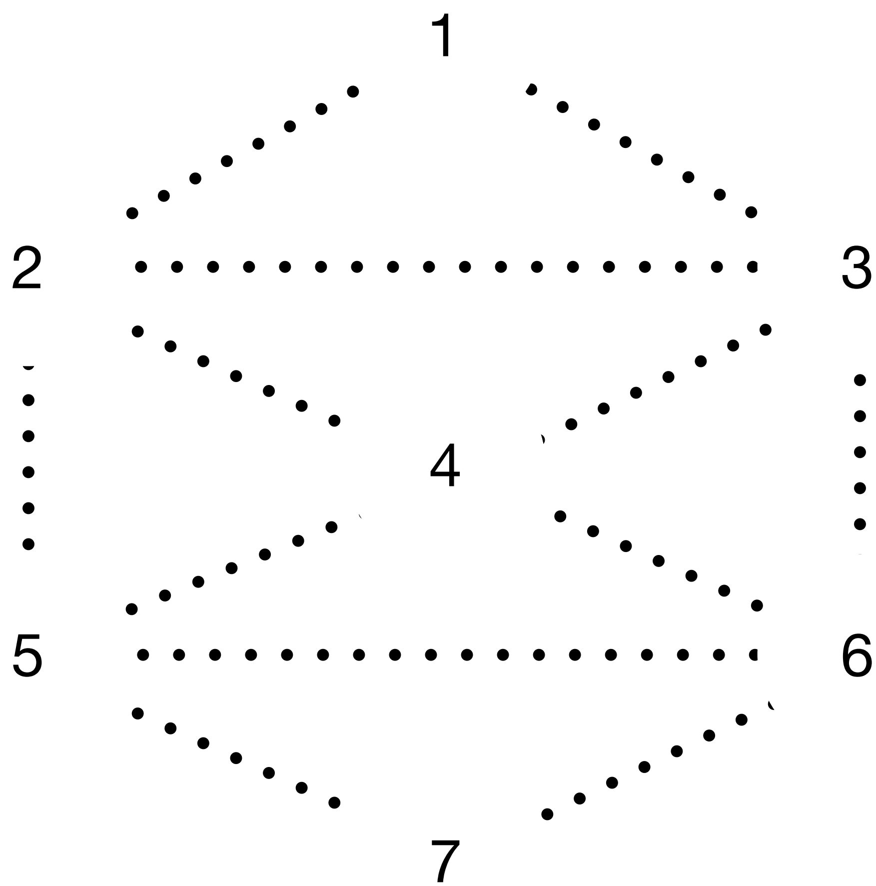


- Does an Euler path exist? **No**
- Nodes with an odd degree must either be the start or end of the path
- Only one node in the Königsberg graph has odd degree; the path cannot exist
- What about an Euler circuit?

Finding an Euler Circuit

- Run a partial DFS; search down a path until you need to backtrack (mark edges instead of nodes)
- At this point, you will have found a circuit
- Find first node along the circuit that has unvisited edges; run a DFS starting with that edge
- Splice the new circuit into the main circuit, repeat until all edges are visited

Euler Circuit Example



Euler Circuit Running Time

- All our DFS's will visit each edge once, so at least O(|E|)
- Must use a linked list for efficient splicing of path, so searching for a vertex with unused edge can be expensive
- but cleverly saving the last scanned edge in each adjacency list can prevent having to check edges more than once, so also O(|E|)

Hamiltonian Cycle

- Now that we know how to find Euler circuits efficiently, can we find Hamiltonian Cycles?
- Hamiltonian cycle - path that visits each *node* once, starts and ends on same node

Reading

- Weiss 9.5 (MST - today's material)
- Weiss 9.6 (DFS - today's material)
- Weiss 9.7 (P vs. NP - Monday)