Announcements

- Homework 5 is out:
  - You can use adjacency lists if you prefer
Review

- Extensions of Dijkstra’s Algorithm
- Critical Path Analysis
- All Pairs Shortest Path (Floyd-Warshall)
- Maximum Flow
- Floyd-Fulkerson Algorithm
Today’s Plan

- Minimum Spanning Tree
- Prim’s Algorithm
- Kruskal’s Algorithm
- Depth first search
- Euler Paths
Minimum Spanning Tree

Problem definition

- Given connected graph $G$, find the connected, acyclic subgraph $T$ with minimum edge weight

- A tree that includes every node is called a **spanning tree**

- The method to find the MST is another example of a greedy algorithm
Motivation for Greed

- Consider any spanning tree
- Adding another edge to the tree creates exactly one cycle
- Removing an edge from that cycle restores the tree structure
Prim’s Algorithm

- Grow the tree like Dijkstra’s Algorithm
- Dijkstra’s: grow the set of vertices to which we know the shortest path
- Prim’s: grow the set of vertices we have added to the minimum tree
- Store shortest edge $D[]$ from each node to tree
Prim’s Algorithm

- Start with a single node tree, set distance of adjacent nodes to edge weights, infinite elsewhere

- Repeat until all nodes are in tree:
  - Add the node $v$ with shortest known distance
  - Update distances of adjacent nodes $w$:
    \[ D[w] = \min(D[w], \text{weight}(v, w)) \]
Prim’s Example
Prim’s Example
Prim’s Example

9 3 9
5 7 3
9 4 6
∞ 5 6
∞ ∞ 5
∞ ∞ ∞
Prim’s Example
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Prim’s Example
Prim’s Example
Prim’s Example

Diagram of a network with edge weights representing distances or costs.
Implementation Details

- Store “previous node” like Dijkstra’s Algorithm; backtrack to construct tree after completion.

- Of course, use a priority queue to keep track of edge weights. Either
  - keep track of nodes inside heap & decreaseKey
  - or just add a new copy of the node when key decreases, and call deleteMin until you see a node not in the tree.
Prim’s Algorithm

Justification

- At any point, we can consider the set of nodes in the tree $T$ and the set outside the tree $Q$.
- Whatever the MST structure of the nodes in $Q$, at least one edge must connect the MSTs of $T$ and $Q$.
- The greedy edge is just as good structurally as any other edge, and has minimum weight.
Prim’s Running Time

- Each stage requires one deleteMin $O(\log |V|)$, and there are exactly $|V|$ stages.
- We update keys for each edge, updating the key costs $O(\log |V|)$ (either an insert or a decreaseKey).
- Total time: $O(|V| \log |V| + |E| \log |V|) = O(|E| \log |V|)$
Kruskal’s Algorithm

- Somewhat simpler conceptually, but more challenging to implement
- Algorithm: repeatedly add the shortest edge that does not cause a cycle until no such edges exist
- Each added edge performs a union on two trees; perform unions until there is only one tree
- Need special ADT for unions (Disjoint Set... we’ll cover it later)
Kruskal’s Example
Kruskal’s Example
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Kruskal’s Example
Kruskal’s Example
Kruskal’s Example
Kruskal’s Justification

- At each stage, the greedy edge $e$ connects two nodes $v$ and $w$
- Eventually those two nodes must be connected;
  - we must add an edge to connect trees including $v$ and $w$
- We can always use $e$ to connect $v$ and $w$, which must have less weight since it's the greedy choice
Kruskal’s Running Time

- First, buildHeap costs $O(|E|)$
- Each edge, need to check if it creates a cycle (costs $O(\log V)$)
- In the worst case, we have to call $|E|$ deleteMins
- Total running time $O(|E| \log |E|)$; but $|E| \leq |V|^2$

\[
O(|E| \log |V|^2) = O(2|E| \log |V|) = O(|E| \log |V|)
\]
MST Wrapup

- Connect all nodes in graph using minimum weight tree
- Two greedy algorithms:
  - Prim’s: similar to Dijkstra’s. Easier to code
  - Kruskal’s: easy on paper
Depth First Search

- Level-order $\leftrightarrow$ Breadth-first Search
- Preorder $\leftrightarrow$ Depth-first Search

- Visit vertex $v$, then recursively visit $v$’s neighbors

- To avoid visiting nodes more than once in a cyclic graph, mark visited nodes,

- and only recurse on unmarked nodes
The Seven Bridges of Königsberg

Königsburg Bridge Problem: can one walk across the seven bridges and never cross the same bridge twice?

http://math.dartmouth.edu/~euler/docs/originals/E053.pdf
The Seven Bridges of Königsberg

Königsburg Bridge Problem: can one walk across the seven bridges and never cross the same bridge twice?

Euler solved the problem by inventing graph theory

http://math.dartmouth.edu/~euler/docs/originals/E053.pdf
Euler Paths and Circuits

- Euler path – a (possibly cyclic) path that crosses each edge exactly once
- Euler circuit - an Euler path that starts and ends on the same node
Euler’s Proof

- Does an Euler path exist? **No**
- Nodes with an odd degree must either be the start or end of the path
- Only one node in the Königsberg graph has odd degree; the path cannot exist
- What about an Euler circuit?
Finding an Euler Circuit

- Run a partial DFS; search down a path until you need to backtrack (mark edges instead of nodes)
- At this point, you will have found a circuit
- Find first node along the circuit that has unvisited edges; run a DFS starting with that edge
- Splice the new circuit into the main circuit, repeat until all edges are visited
Euler Circuit Example
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Euler Circuit Running Time

- All our DFS's will visit each edge once, so at least $O(|E|)$

- Must use a linked list for efficient splicing of path, so searching for a vertex with unused edge can be expensive

- but cleverly saving the last scanned edge in each adjacency list can prevent having to check edges more than once, so also $O(|E|)$
Hamiltonian Cycle

- Now that we know how to find Euler circuits efficiently, can we find Hamiltonian Cycles?

- Hamiltonian cycle - path that visits each node once, starts and ends on same node
Reading

- Weiss 9.5 (MST - today’s material)
- Weiss 9.6 (DFS - today’s material)
- Weiss 9.7 (P vs. NP - Monday)