Announcements

- Homework 4 due next class
- Huffman compression must handle any characters in dictionary.txt
- Spell checker can ignore case
Review

- Rehashing
- String hash function example
- Graphs
  - Terminology and properties
  - Implementation
Today’s Plan

- Topological Sort
- Shortest Path
  - Unweighted version
  - Weighted version
Implementation

- Option 1:
  - Store all nodes in an indexed list
  - Represent edges with adjacency matrix

- Option 2:
  - Explicitly store adjacency lists
Adjacency Matrices

- 2d-array $A$ of boolean variables
- $A[i][j]$ is true when node $i$ is adjacent to node $j$
- If graph is undirected, $A$ is symmetric

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Adjacency Lists

- Each node stores references to its neighbors

```
  1  2  3
  2  1  4
  3  1  4
  4  2  3  5
  5  4
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Math Notation for Graphs

- Set Notation:
  - \( v \in V \) (v is in V)
  - \( U \cup V \) (union)
  - \( U \cap V \) (intersection)
  - \( U \subset V \) (U is a subset of V)

- \( G = \{V, E\} \)
- \( G \) is the graph
- \( V \) is set of vertices
- \( E \) is set of edges
- \( (v_i, v_j) \in E \)
- \( |V| = N = \) size of \( V \)
Topological Sort

- Problem definition:
  
  - Given a directed acyclic graph $G$, order the nodes such that for each edge $(v_i, v_j) \in E$, $v_i$ is before $v_j$ in the ordering.

  - e.g., scheduling errands when some tasks depend on other tasks being completed.
Topological Sort Ex.

- Buy Groceries
- Look up recipe online
- Buy Stamps
- Fix Computer
- Pay Taxes
- Go to ATM
- Mail
- Mail recipe to Grandma
- Mail Postcard
- Mail Tax Form
- Cook Dinner
- Look up recipe online
- Mail recipe to Grandma
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- Cook Dinner
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Topological Sort
Naïve Algorithm

- **Degree** means # of edges, **indegree** means # of incoming edges

1. Compute the **indegree** of all nodes
2. Print any node with indegree 0
3. Remove the node we just printed. Go to 1.

Which nodes’ indegrees change?
Topological Sort
Better Algorithm

1. Compute all indegrees
2. Put all indegree 0 nodes into a Collection
3. Print and remove a node from Collection
4. Decrement indegrees of the node’s neighbors.
5. If any neighbor has indegree 0, place in Collection. Go to 3.
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- **Buy Groceries**
- **Cook Dinner**
- **Look up recipe online**
- **Buy Stamps**
- **Mail recipe to Grandma**
- **Mail Postcard**
- **Mail Tax Form**
- **Go to ATM**
- **Fix Computer**

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**Diagram:**

- **Buy Groceries**
- **Cook Dinner**
- **Look up recipe online**
- **Buy Stamps**
- **Mail recipe to Grandma**
- **Mail Postcard**
- **Mail Tax Form**
- **Fix Computer**
- **Go to ATM**

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**Flowchart Diagram:***
- **Go to ATM**
- **Buy Groceries**
- **Look up recipe online**
- **Buy Stamps**
- **Mail recipe to Grandma**
- **Cook Dinner**
- **Mail Postcard**
- **Mail Tax Form**
- **Fix Computer**

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Topological Sort

Running time

- Initial indegree computation: $O(|E|)$
  - Unless we update indegree as we build graph
- $|V|$ nodes must be enqueued/dequeued
- Dequeue requires operation for outgoing edges
- Each edge is used, but never repeated
- Total running time $O(|V| + |E|)$
Shortest Path

* Given $G = (V,E)$, and a node $s \in V$, find the shortest (weighted) path from $s$ to every other vertex in $G$.

* Motivating example: subway travel
  * Nodes are junctions, transfer locations
  * Edge weights are estimated time of travel
Approximate MTA Express Stop Subgraph

* A few inaccuracies (don’t use this to plan any trips)
Breadth First Search

- Like a level-order traversal
- Find all adjacent nodes (level 1)
- Find new nodes adjacent to level 1 nodes (level 2)
- ... and so on
- We can implement this with a queue
Unweighted Shortest Path Algorithm

Set node s’ distance to 0 and enqueue s.

Then repeat the following:

Dequeue node v. For unset neighbor u:

set neighbor u’s distance to v’s distance +1
mark that we reached v from u
enqueue u
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Weighted Shortest Path

- The problem becomes more difficult when edges have different weights
- Weights represent different costs on using that edge
- Standard algorithm is Dijkstra’s Algorithm
Dijkstra’s Algorithm

- Keep distance overestimates $D(v)$ for each node $v$
  (all non-source nodes are initially infinite)

1. Choose node $v$ with smallest unknown distance

2. Declare that $v$’s shortest distance is known

3. Update distance estimates for neighbors
Updating Distances

- For each of \( v \)'s neighbors, \( w \),
- if \( \min(D(v) + \text{weight}(v,w), D(w)) \)
- i.e., update \( D(w) \) if the path going through \( v \) is cheaper than the best path so far to \( w \)
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Dijkstra’s Algorithm Analysis

- First, convince ourselves that the algorithm works.
- At each stage, we have a set of nodes whose shortest paths we know.
- In the base case, the set is the source node.
- Inductive step: if we have a correct set, is greedily adding the shortest neighbor correct?
Proof by Contradiction (Sketch)

- Contradiction: Dijkstra’s finds a shortest path to node \( w \) through \( v \), but there exists an even shorter path.

- This shorter path must pass from inside our known set to outside.

- Call the 1st node in cheaper path outside our set \( u \).

- The path to \( u \) must be shorter than the path to \( w \).

- But then we would have chosen \( u \) instead.
Computational Cost

- Keep a priority queue of all unknown nodes
- Each stage requires a `deleteMin`, and then some `decreaseKeys` (the # of neighbors of node)
- We call `decreaseKey` once per edge, we call `deleteMin` once per vertex
- Both operations are $O(\log |V|)$
- Total cost: $O(|E| \log |V| + |V| \log |V|) = O(|E| \log |V|)$
Reading

- Weiss Section 9.3 (today’s material)
- Weiss Section 9.4 (Monday’s material)