Data Structures and Algorithms

Session 16. March 25, 2009

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Announcements

- Homework 4 up on website
- Hw3 grades coming tomorrow
- Thanks for feedback on midterm evaluations
- I have old hw's and midterms in my office; stop by after class or let's set up a time for pickup

Review

- Midterm Solutions
- Huffman Coding Trees
  - Create forest of all characters weighted by frequency
  - Merge least weight trees until 1 tree left
- (Unfinished) proof sketch of optimality

Today's Plan

- Finish Huffman optimality proof sketch
- Hash Tables ADT
  - Definition and Implementation

Huffman Details

- We can manage the forest with a priority queue:
- **buildHeap** first,
  - find the least weight trees with 2 **deleteMin**s,
  - after merging, **insert** back to heap.
- In practice, also have to store coding tree, but the payoff comes when we compress larger strings

Optimality of Huffman

- Induction: Suppose Huffman tree is optimal for **N** characters. What about **N+1** characters?
  - Lemma 1: Optimal tree is full
  - Lemma 2: the 2 least frequent characters are at the deepest level in optimal tree
  - Lemma 3: Swapping characters at same depth doesn’t affect optimality

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  - Lemma 3: Swapping characters at same depth doesn’t affect optimality
- Lemma 4: An optimal tree exists where the least frequent characters are siblings at deepest level.

Optimality of Huffman

- number of bits of an encoding is $B(T) = \sum_{i=1}^{N+1} F_i D_i$
- F is the frequency of the character, D is the depth in the tree (the number of bits)
- Create new tree T* by removing least frequent chars and replacing with a meta-character whose frequency is the frequency of both chars,
    - meta-character is one level less deep

Optimality of Huffman

- $B(T) = B(T*) + F_1 + F_2$
- Proof by contradiction: Assume there is a different tree T' that is better than T

$$\begin{aligned} B(T') &< B(T) \\ B(T'*) + F_1 + F_2 &< B(T*) + F_1 + F_2 \\ B(T'*) &< B(T*) \end{aligned}$$

- That is a contradiction because T* has N characters, which means Huffman is optimal via our inductive hypothesis

Optimality of Huffman

- Assuming falseness of inductive **step** produced contradiction to inductive **hypothesis**
- Therefore, if Huffman codes are optimal for **N** characters, they are also for **N+1** characters
- Huffman is obviously optimal for 2 characters
- Huffman codes are optimal $\square$

Hash Table ADT

- **Search tree:**
  findMin, findMax, insert/delete, search
- **Priority Queue:**
  findMin (or max), insert/delete, <u>no</u> search
- **Hash Table:**
  insert/delete, search

Hash Table ADT

- **Search tree:**
  Stores complete order information
- **Priority Queue:**
  Stores <u>in</u>complete order information
- **Hash Table:**
  Stores <u>no</u> order information

Hash Table ADT

- Insert or delete objects by **key**
- Search for objects by **key**
- **No** order information whatsoever
- Ideally O(1) per operation

Implementation

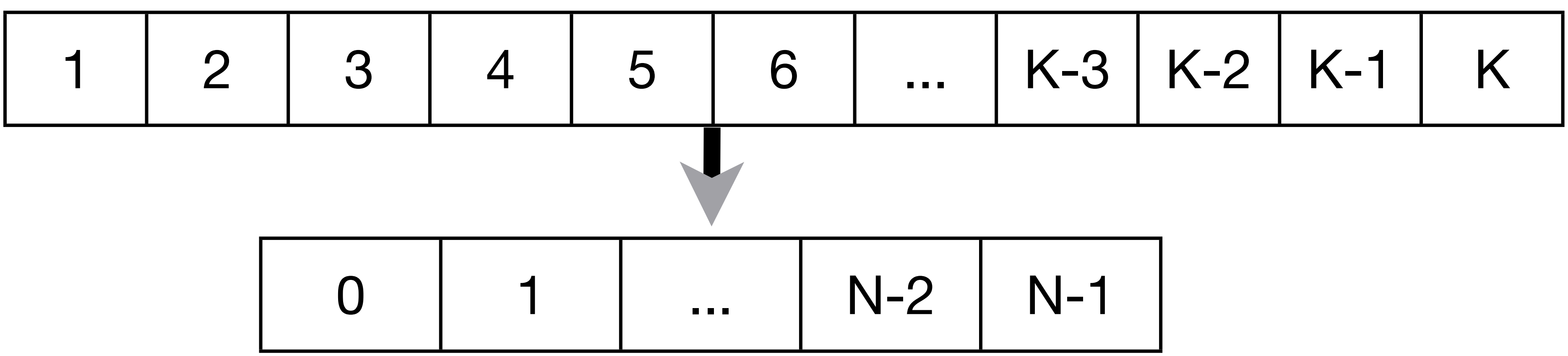
- Suppose we have keys between 1 and K
- Create an array with maxKey entries
- Insert, delete, search are just array operations

1	2	3	4	5	6	...	K-3	K-2	K-1	K

- Obviously too expensive

Hash Functions

- A **hash function** maps any key to a valid array position
  - Array positions range from 0 to N-1
  - Key range possibly unlimited



Hash Functions

- For integer keys, (key mod N) is the simplest hash function
- In general, **any** function that maps from the space of keys to the space of array indices is valid
- but a good hash function spreads the data out evenly in the array;
- A good hash function avoids **collisions**

Collisions

- A **collision** is when two distinct keys map to the same array index
  - e.g., $h(x) = x \bmod 5$
    $h(7) = 2$, $h(12) = 2$
- Choose $h(x)$ to minimize collisions, but collisions are inevitable
- To implement a hash table, we must decide on collision resolution policy

Collision Resolution

- Two basic strategies
  - Strategy 1: Separate Chaining
  - Strategy 2: Probing; lots of variants

Strategy 1: Separate Chaining

- Keep a list at each array entry
  - Insert(x): find h(x), add to list at h(x)
  - Delete(x): find h(x), search list at h(x) for x, delete
  - Search(x): find h(x), search list at h(x)
- We could use a BST or other ADT, but if h(x) is a good hash function, it won’t be worth the overhead

Separate Chaining Average Case

- Load Factor λ = # objects / TableSize
  - Average list length is λ
  - Time to insert = constant, or constant + λ
  - Time to search = constant +λ or constant + $\lambda/2$

Strategy 1: Advantages and Disadvantages

- Advantages:
  - Simple idea
  - Removals are clean *
- Disadvantages:
  - Need 2nd data structure, which causes extra overhead if the hash function is good

Strategy 2: Probing

- If $h(x)$ is occupied, try **$h(x)+f(i)$ mod N** for $i = 1$ until an empty slot is found
- Many ways to choose a good $f(i)$
- Simplest method: Linear Probing
  - $f(i) = i$

Linear Probing Example

- N = 5

- h(x) = x mod 5
- insert 7

		7		

- insert 12

		7	12	

- insert 2

		7	12	2

Primary Clustering

- If there are many collisions, blocks of occupied cells form: **primary clustering**
- Any hash value inside the cluster adds to the end of that cluster
- (a) it becomes more likely that the next hash value will collide with the cluster, and (b) collisions in the cluster get more expensive

Removals

- How do we delete when probing?
- Lazy-deletion: mark as deleted,
    - we can overwrite it if inserting,
    - but we know to keep looking if searching.

Quadratic Probing

- $f(i) = i^2$
- Avoids primary clustering
- Sometimes will never find an empty slot even if table isn't full!
- Luckily, if load factor $\lambda \leq \frac{1}{2}$, guaranteed to find empty slot

Quadratic Probing Example

- N = 7

- h(x) = x mod 7
- insert 9

		9				

- insert 16

		9	16			

- insert 2

		9	16			2

Double Hashing

- If $h_1(x)$ is occupied, probe according to
$$f(i) = i \times h_2(x)$$
- 2nd hash function must never map to 0
- Increments differently depending on the key

Double Hashing Example

- N = 7

- h1(x) = x mod 7,   h2(x) = 5-x mod 5
- insert 9

		9				

- insert 16

		9				16

- insert 2

		7			2	16

Reading

- Homework 4
- Weiss Ch. 5