Data Structures in Java

Session 19

Instructor: Bert Huang

http://www.cs.columbia.edu/~bert/courses/3134

Announcements

- Homework 5 due 11/24

Review

- Minimum Spanning Tree
  - Prim's algorithm: similar to Dijkstra
  - Kruskal's algorithm
- Disjoint Set ADT

Today's Plan

- Review Disjoint Set ADT
- Start Discussion of Sorting
  - Lower bound
  - Breaking the lower bound

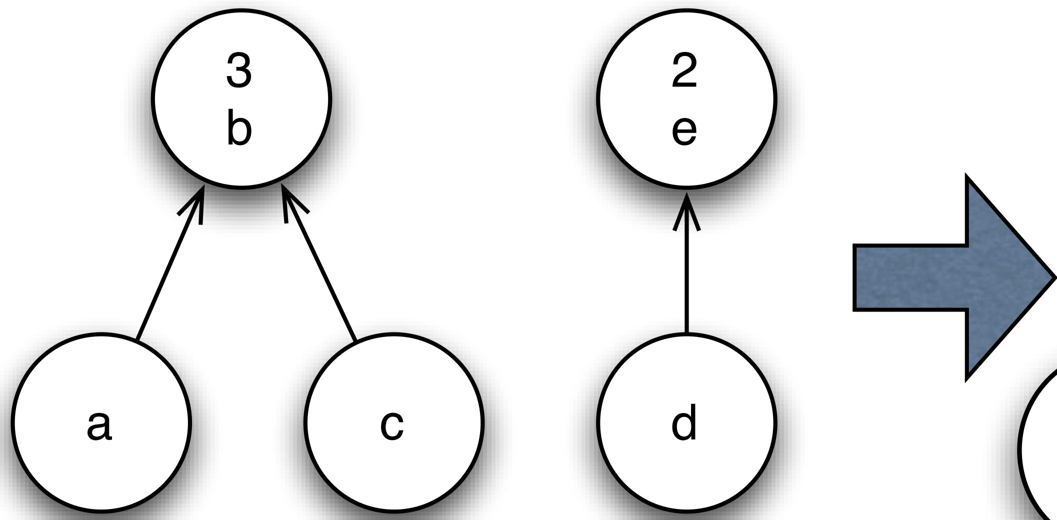
Analysis

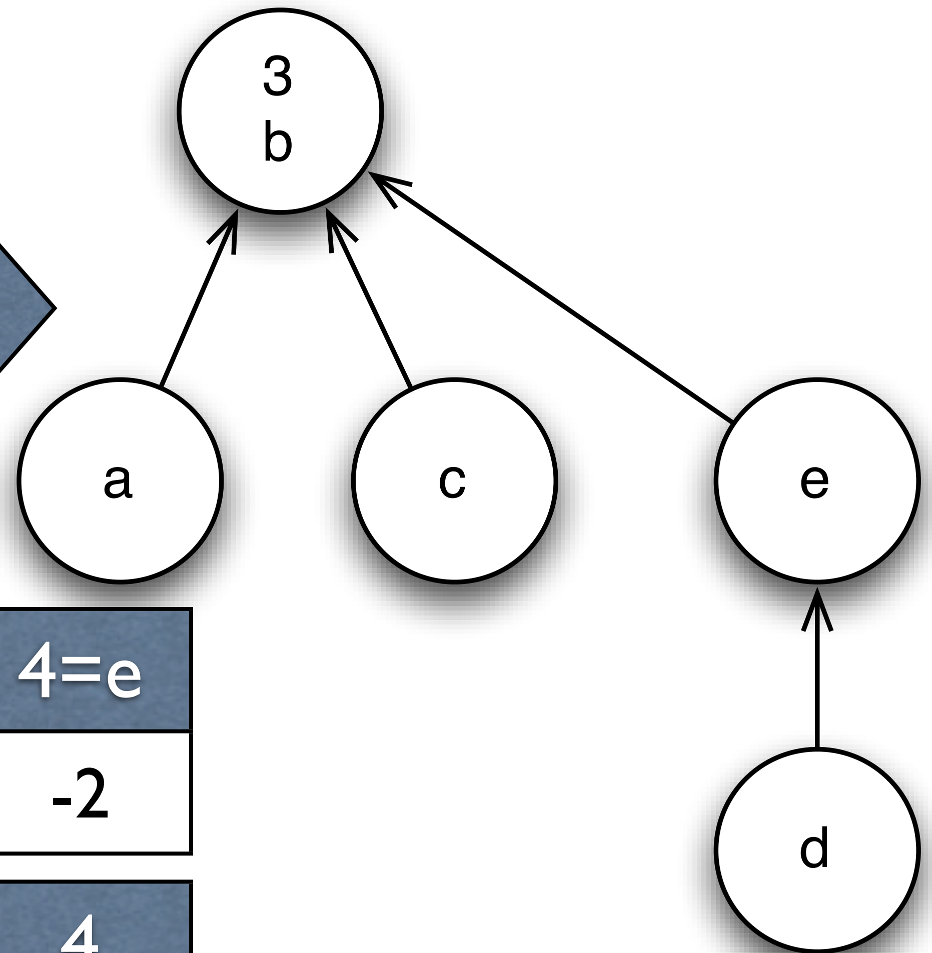
- **find** costs the depth of the node
- **union** costs O(1) after **find**ing the roots
- Both operations depend on the height of the tree
- Since these are general trees, the trees can be arbitrarily shallow

Union by Size

- Claim: if we union by pointing the smaller tree to the larger tree's root, the height is at most log N
- Each union increases the depths of nodes in the smaller trees
- Also puts nodes from the smaller tree into a tree at least twice the size
  - We can only double the size log N times

Union by Size Figure





0=a	1=b	2=c	3=d	4=e
1	-3	1	4	-2

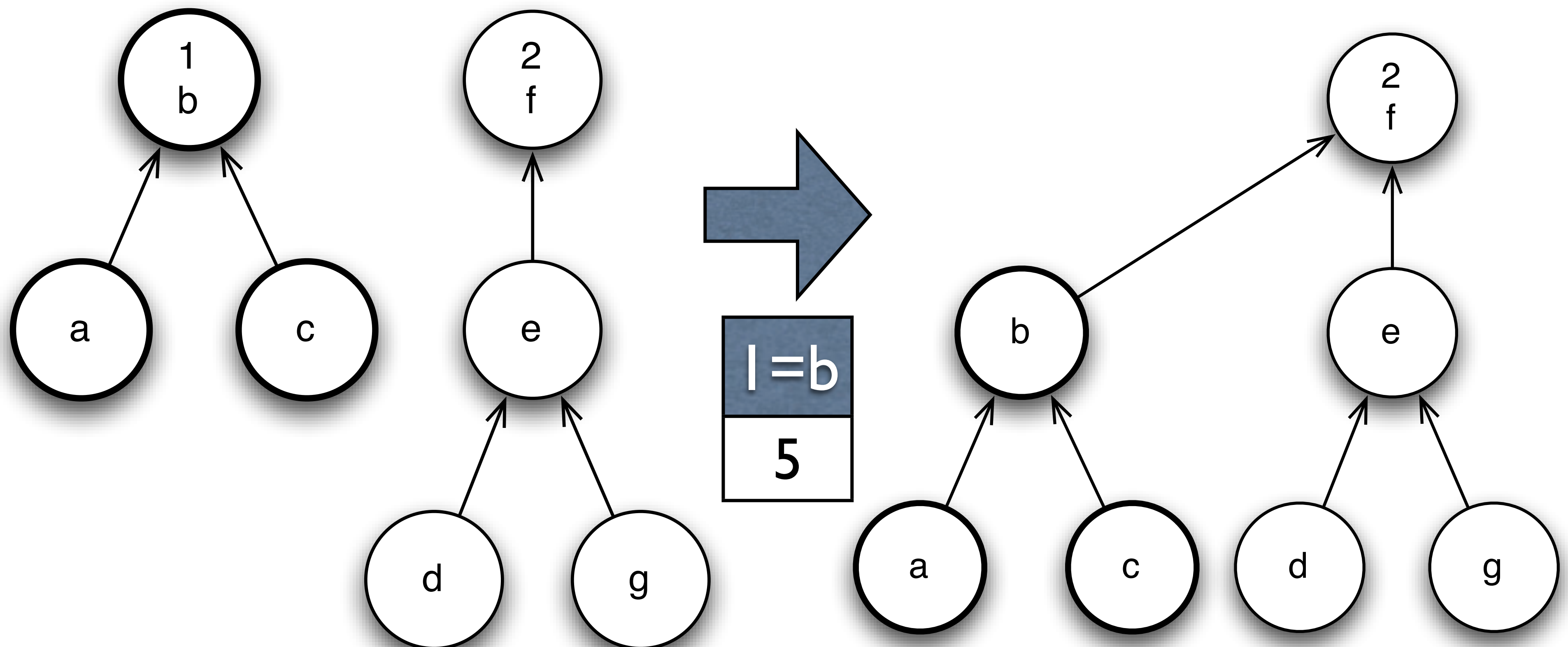
0	1	2	3	4
1	-5	1	4	1

Union by Height

- Similar method, attach the tree with less height to the taller tree
- overall height only increases if trees are equal height

Union by Height Figure

0=a	1=b	2=c	3=d	4=e	5=f	6=g
1	-1	1	4	5	-2	4



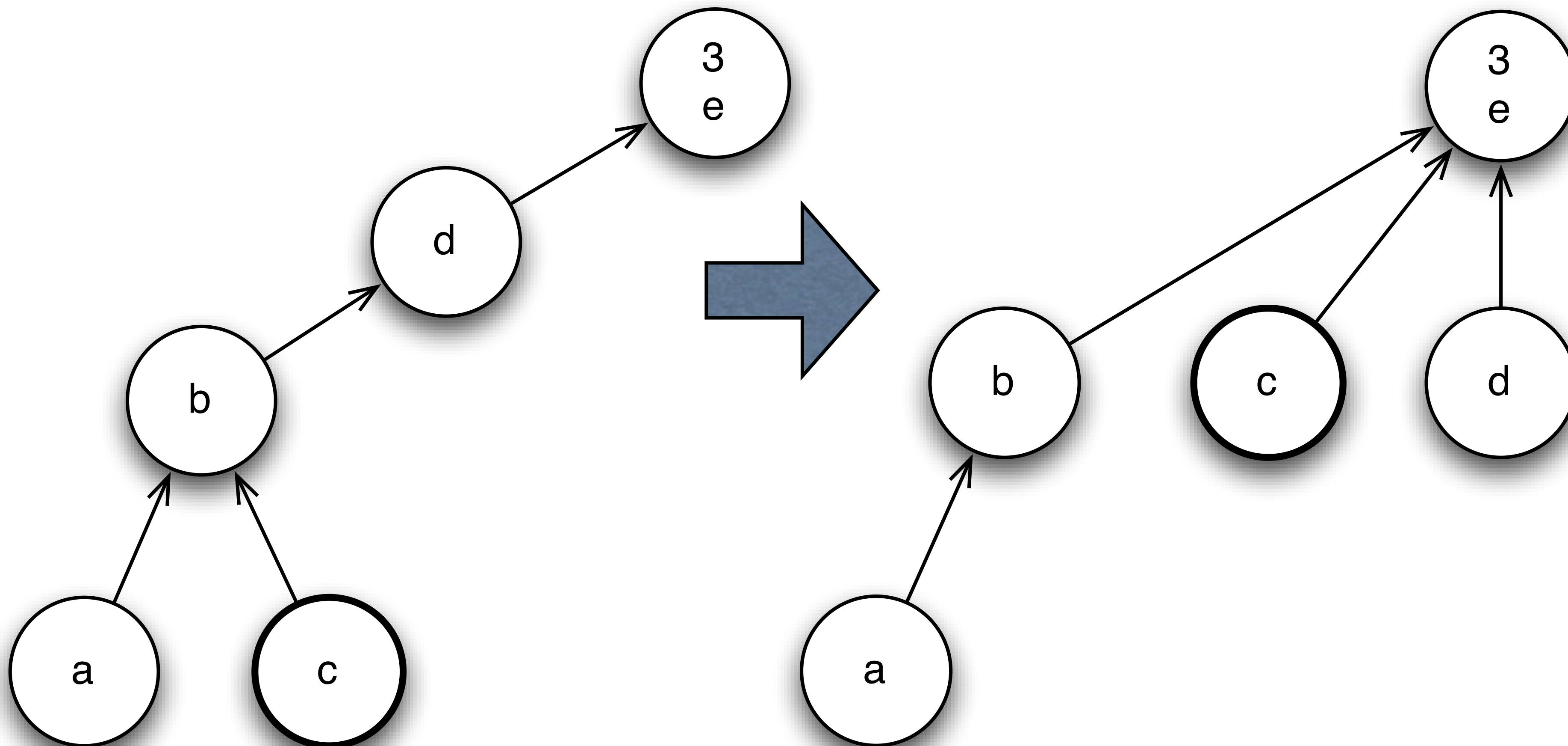
Union by Height proof

- Induction: tree of height **h** has at least 2^h nodes
- Let **T** be tree of height **h** with least nodes possible via union operations
- At last union, **T** must have had height **h-1**, because otherwise, it would have been a smaller tree of height **h**
- Since the height was updated, **T** unioned with another tree of height **h-1**, each had at least 2^{h-1} nodes resulting in at least 2^h nodes for **T**

Path Compression

- Even if we have log N tall trees, we can keep calling **find** on the deepest node repeatedly, costing O(M log N) for M operations
- Additionally, we will perform **path compression** during each **find** call
  - Point every node along the find path to root

Path Compression Figure



0=a	1=b	2=c	3=d	4=e
1	3	1	4	-3

0=a	1=b	2=c	3=d	4=e
1	4	4	4	-3

Union by Rank

- Path compression messes up union-by-height because we reduce the height when we compress
- We could fix the height, but this turns out to gain little, and costs **find** operations more
- Instead, rename to **union by rank**, where **rank** is just an overestimate of height
- Since heights change less often than sizes, rank/height is usually the cheaper choice

Worst Case Bound

- Any sequence of $M = \Omega(N)$ operations will cost **O(M log* N)** running time
- log* N is the number of times the logarithm needs to be applied to N until the result is ≤ 1
- So for all realistic intents, each operation is amortized constant time

Note about Kruskal’s

- With this bound, Kruskal’s algorithm needs N-1 unions, so it should cost almost linear time to perform unions
- Unfortunately the algorithm is still dominated by heap deleteMin calls, so asymptotic running time is still O(E log V)

Sorting

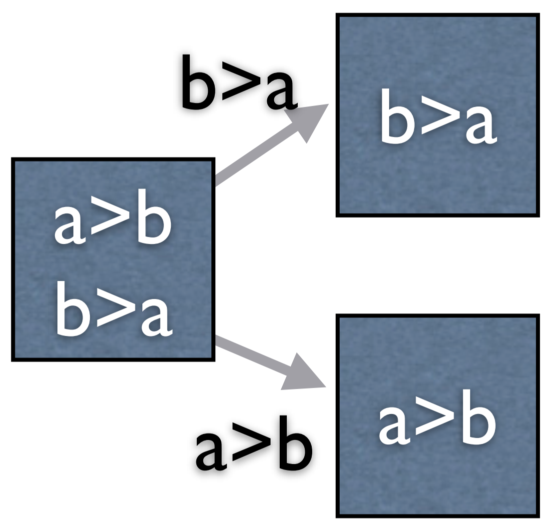
- Given array A of size N, reorder A so its elements are in order.
  - "In order" with respect to a consistent comparison function

The Bad News

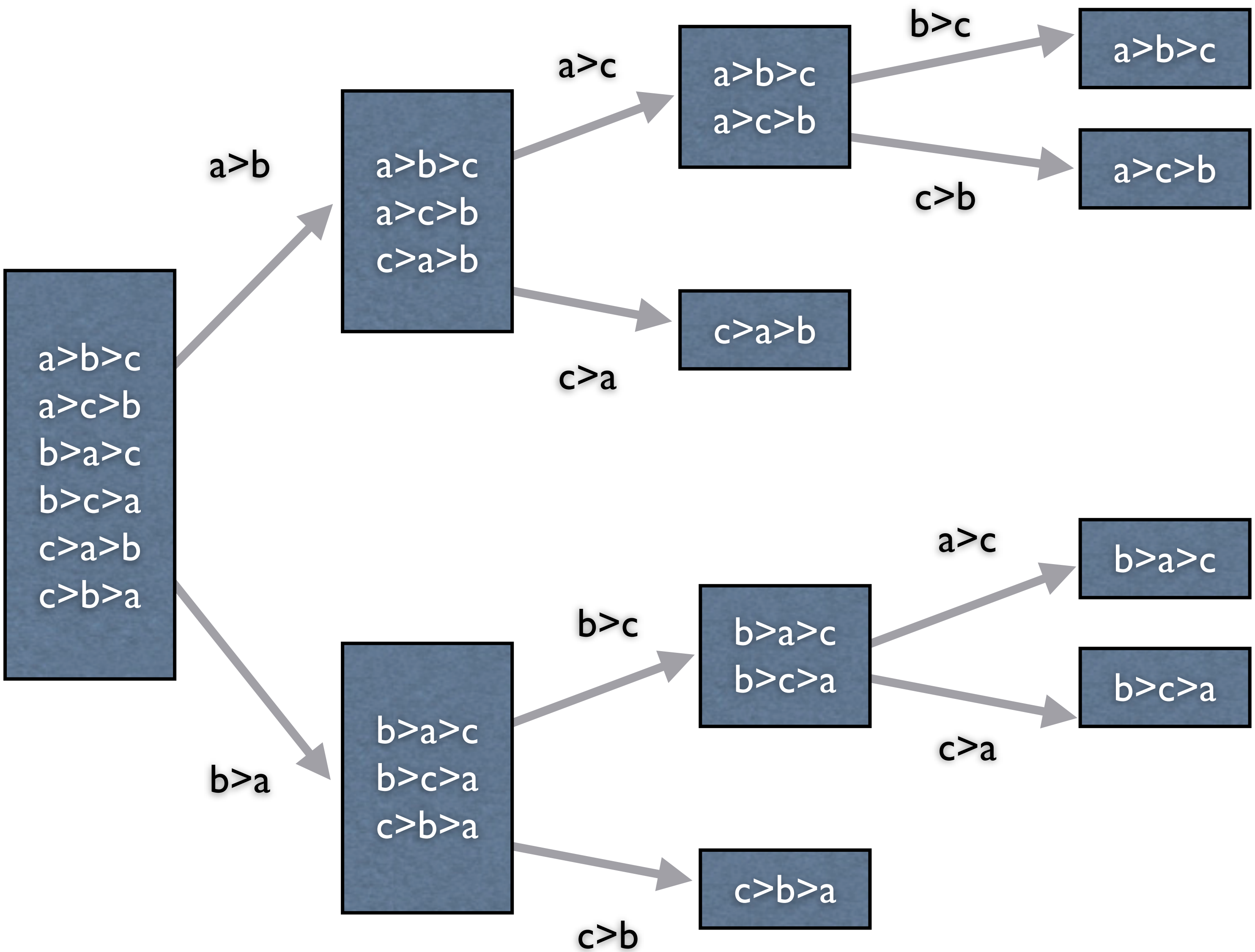
- Sorting algorithms typically compare two elements and branch according to the result of comparison
- **Theorem**: An algorithm that branches from the result of pairwise comparisons must use $\Omega(N \log N)$ operations to sort worst-case input
- Proof. Consider the decision tree

Comparison Sort Decision Tree: N=2

- Each node in this decision tree represents a state
- Move to child states after any branch
- Consider the possible orderings at each state



Decision Tree: N=3



Lower Bound Proof

- The worst case is the deepest leaf; the height
- Lemma 7.1: Let **T** be a binary tree of depth **d**. Then **T** has at most 2^d leaves
- Proof. By induction.
  Base case: d = 0, one leaf
  - Otherwise, we have root and left/right subtrees of depth at most **d-1**. Each has at most 2^{d-1} leaves

Lower Bound Proof

- Lemma 7.1: Let **T** be a binary tree of depth **d**. Then **T** has at most 2^d leaves
- Lemma 7.2: A binary tree with **L** leaves must have [height] at least $\lceil \log L \rceil$
- Theorem proof. There are N! leaves in the binary decision tree for sorting. Therefore, the deepest node is at depth $\log(N!)$

Lower Bound Proof

$$\log(N!)$$

$$\begin{aligned}
&= \log(N(N-1)(N-1)\dots(2)(1)) \\
&= \log N + \log(N-1) + \log(N-2) + \dots + \log 2 + \log 1 \\
&\geq \log N + \log(N-1) + \log(N-2) + \dots + \log(N/2) \\
&\geq \frac{N}{2} \log \frac{N}{2} \\
&\geq \frac{N}{2} \log N - \frac{N}{2} \\
&= \Omega(N \log N)
\end{aligned}$$

Comparison Sort Lower Bound

- Decision tree analysis provides nice mechanism for lower bound
- However, the bound only allows pairwise comparisons.
- We've already learned a data structure that beats the bound
  - What is it?

Trie Running Time

- Insert items into trie then preorder traversal
- Each insert costs **O(k)**, for length of word **k**
- **N** inserts cost **O(Nk)**
- Preorder traversal costs **O(Nk)**, because the worst case trie has each word as a leaf of a disjoint path of length **k**
  - This is a very degenerate case

Counting Sort

- Another simple sort for integer inputs
- 1. Treat integers as array indices (subtract min)
- 2. Insert items into array indices
- 3. Read array in order, skipping empty entries
- 4. Laugh at comparison sort algorithms

Bucket Sort

- Like Counting Sort, but less wasteful in space
- Split the input space into **k** buckets
- Put input items into appropriate buckets
- Sort the buckets using favorite sorting algorithm

Radix Sort

- TrieSort and CountingSort are forms of Radix Sort
- Radix Sort sorts by looking at one digit at a time
- We can start with the least significant digit or the most significant digit
  - least significant digit first provides a **stable** sort
  - tries use most significant, so let's look at least...

Radix Sort with Least Significant Digit

- BucketSort according to the least significant digit
- Repeat: BucketSort contents of each multi-item bucket according to the next least significant digit
- Running time: **O(Nk)** for maximum of **k** digits
- Space: **O(Nk)**

Comparison Sorting

- Nevertheless, comparison-based sorting is much more general
- Well-studied problem, lots of different algorithms with various tradeoffs
- We'll examine some of the famous algorithms

Reading

- Disj. Sets:
  Weiss Ch. 8 (skim proof in 8.6)
- Sorting:
  Weiss Section 7.8 (lower bound)
  Rest of Section 7 for next two classes