Data Structures in Java

Session 18

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Announcements

- Homework 5 posted, due 11/24
- Old homeworks, midterm exams

Review

- Shortest Path algorithms
  - Breadth first search
  - Dijkstra’s Algorithm
  - All-Pairs Shortest Path

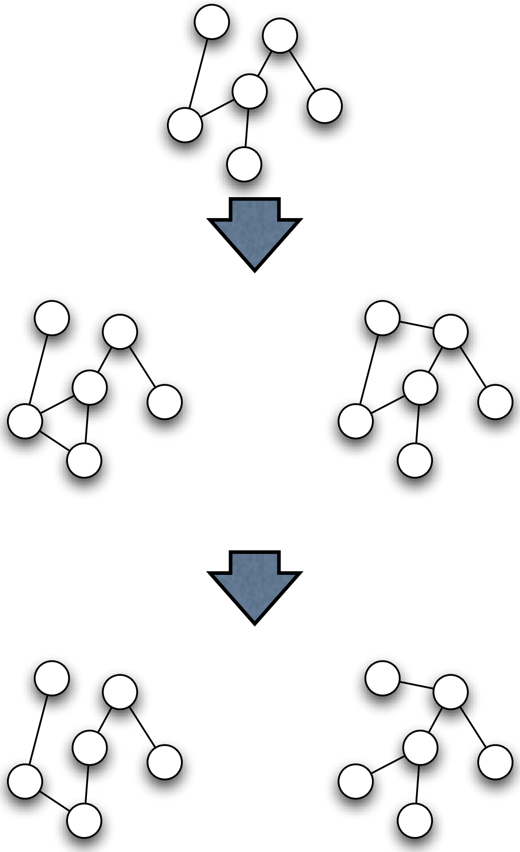
Today’s Plan

- Minimum Spanning Tree
    - Prim’s Algorithm
    - Kruskal’s Algorithm
- Disjoint Sets

Minimum Spanning Tree Problem Definition

- Given connected graph **G**, find the connected, acyclic subgraph **T** with minimum edge weight
    - A tree that includes every node is called a **spanning tree**
- The method to find the MST is another example of a greedy algorithm

Motivation for Greed



- Consider any spanning tree
- Adding another edge to the tree creates exactly one cycle
- Removing an edge from that cycle restores the tree structure

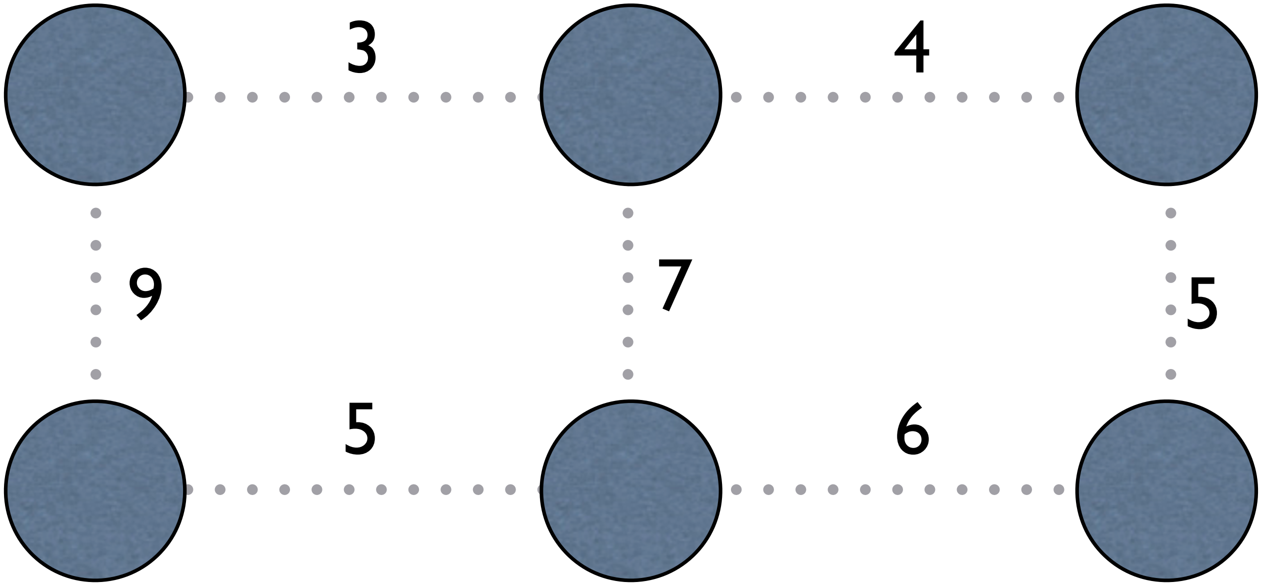
Prim’s Algorithm

- Grow the tree like Dijkstra’s Algorithm
- Dijkstra’s: grow the set of vertices to which we know the shortest path
- Prim’s: grow the set of vertices we have added to the minimum tree
- Store shortest edge **D**[ ] from each node to tree

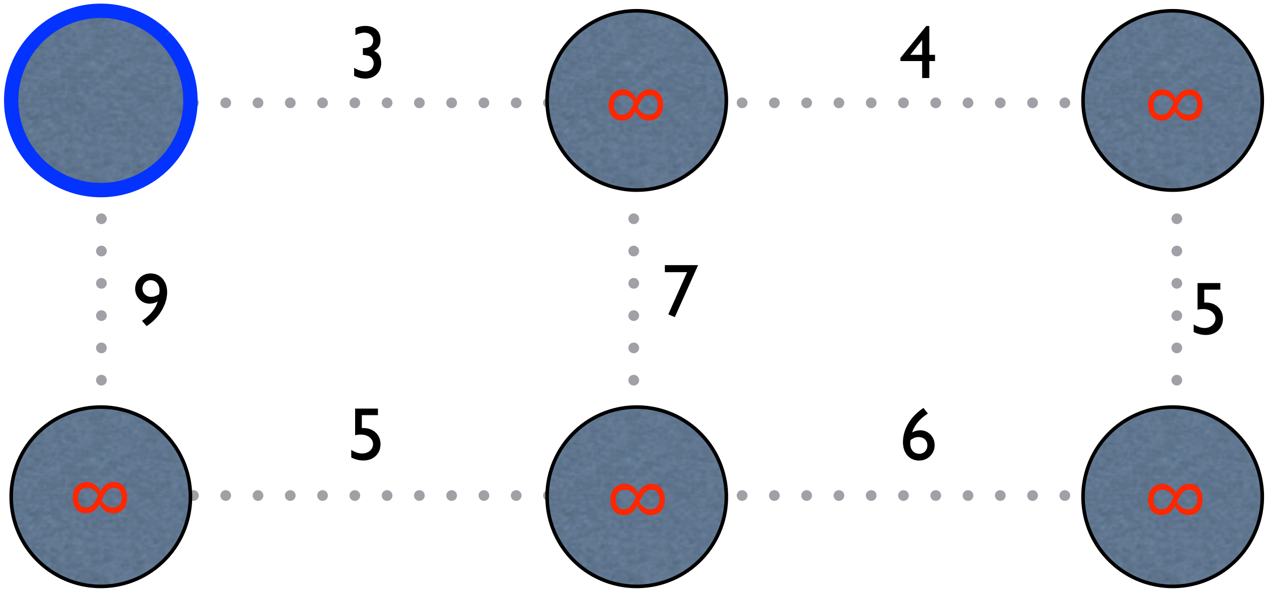
Prim’s Algorithm

- Start with a single node tree, set distance of adjacent nodes to edge weights, infinite elsewhere
- Repeat until all nodes are in tree:
  - Add the node **v** with shortest known distance
  - Update distances of adjacent nodes **w**: **D**[w] = min(**D**[w], weight(**v**,**w**))

Prim’s Example



Prim’s Example



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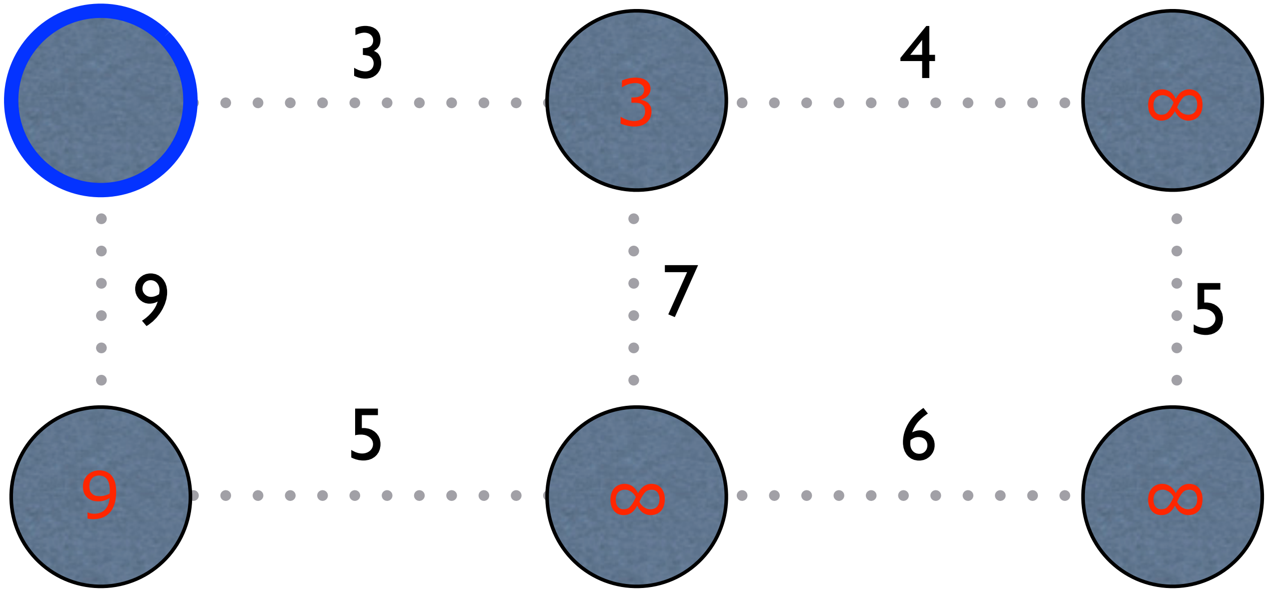

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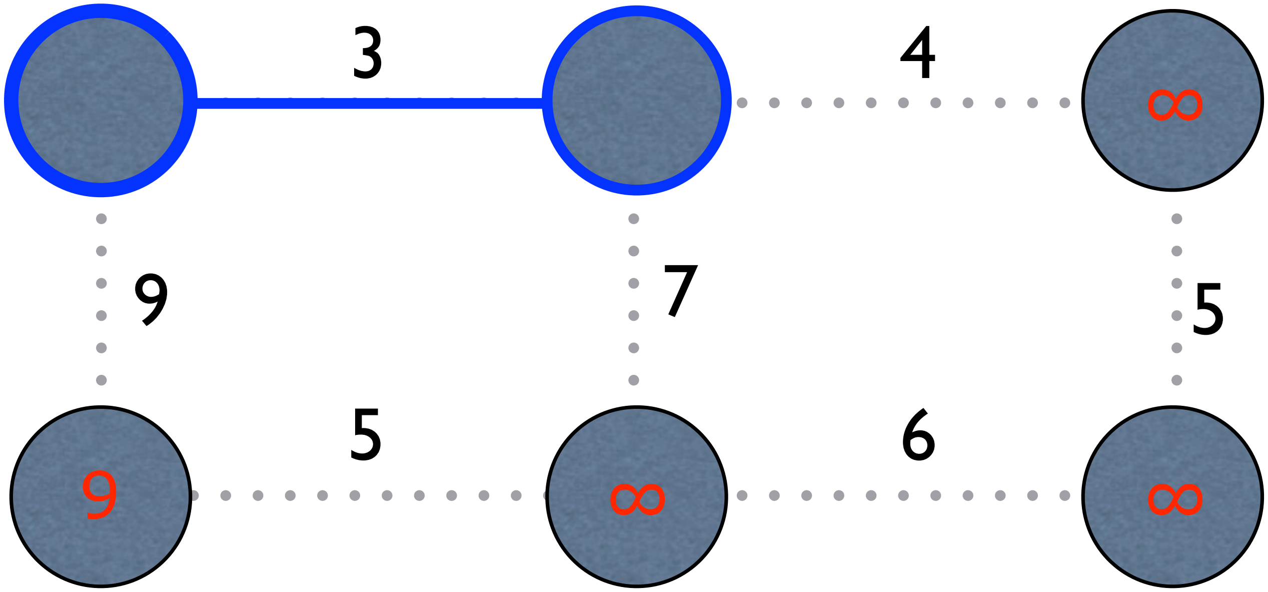

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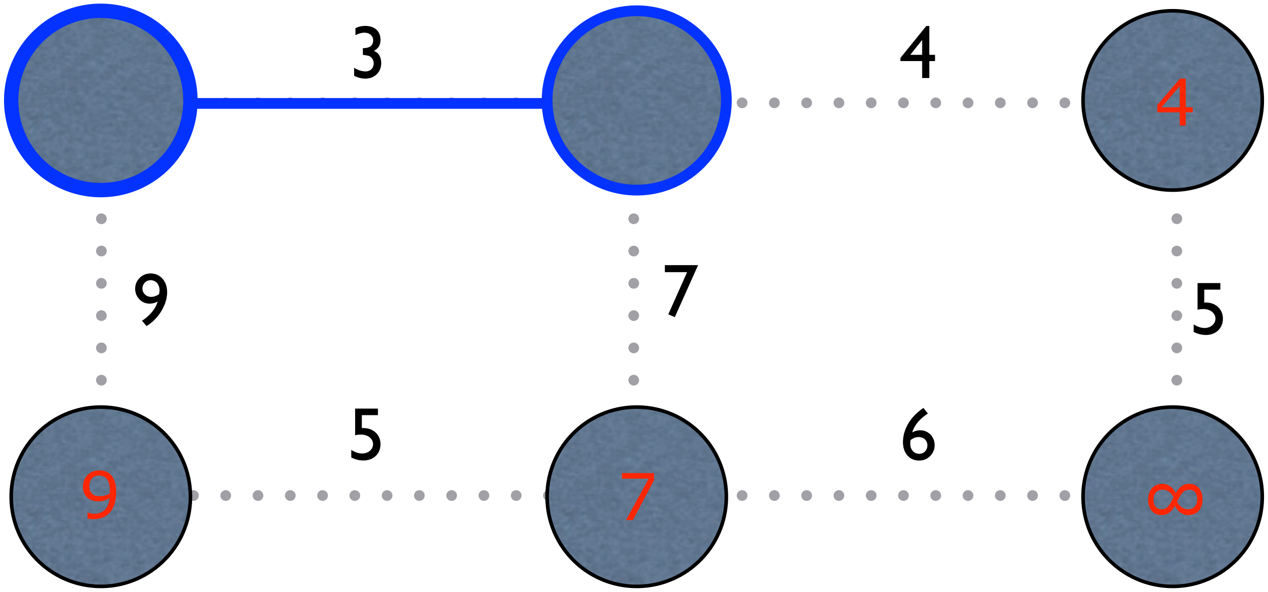

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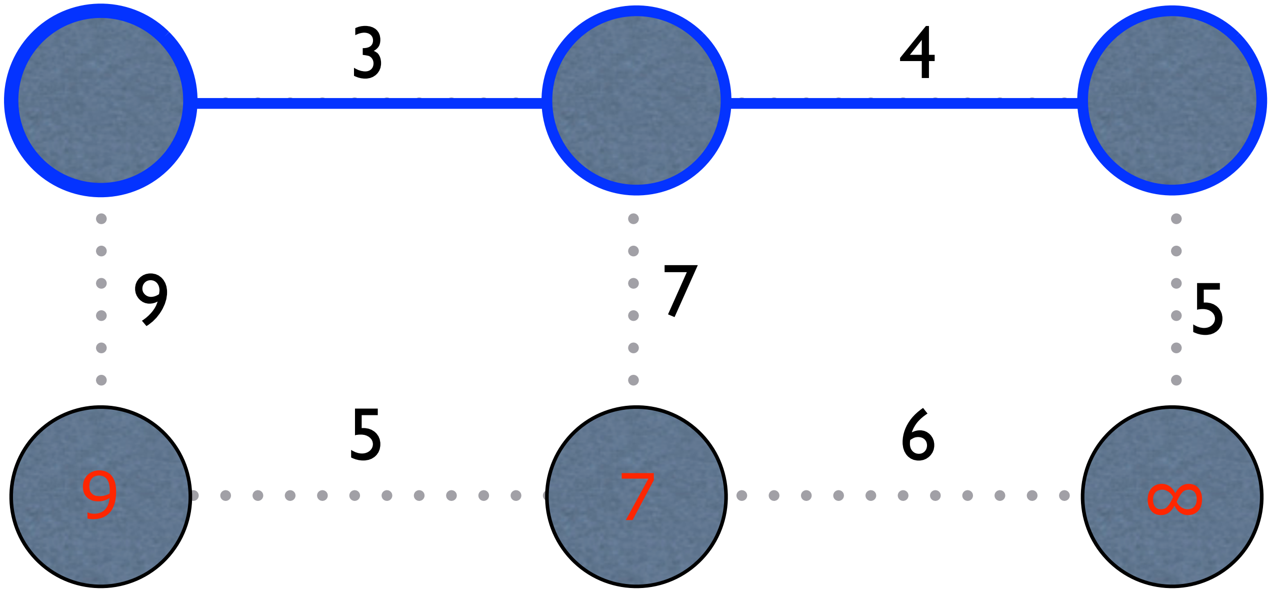

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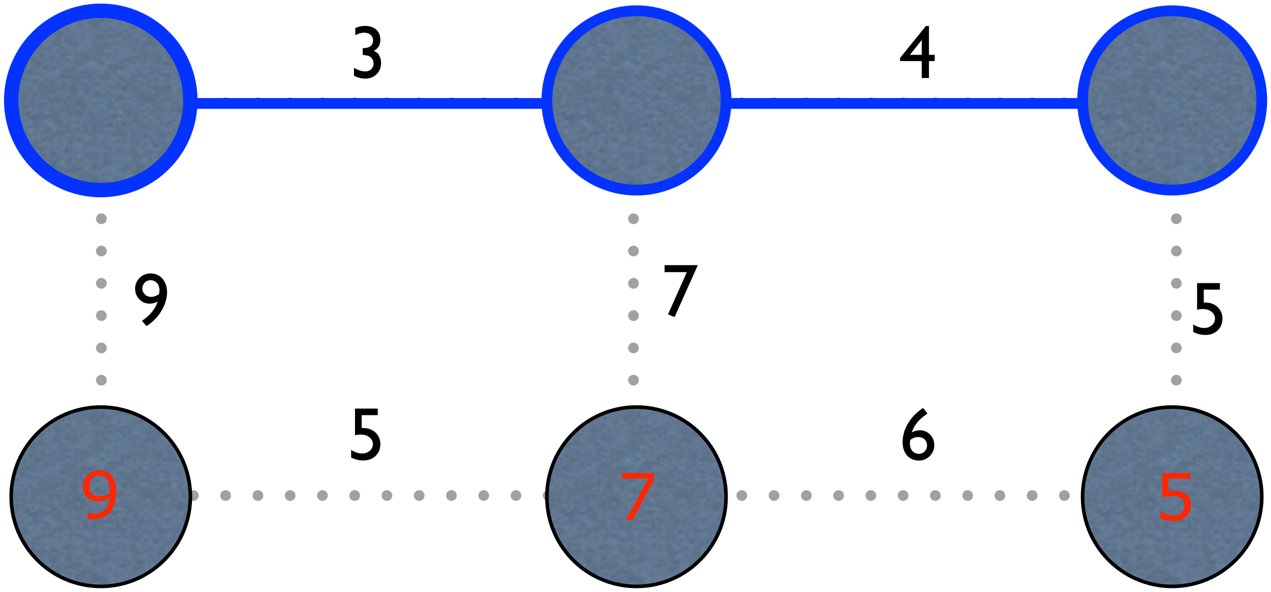

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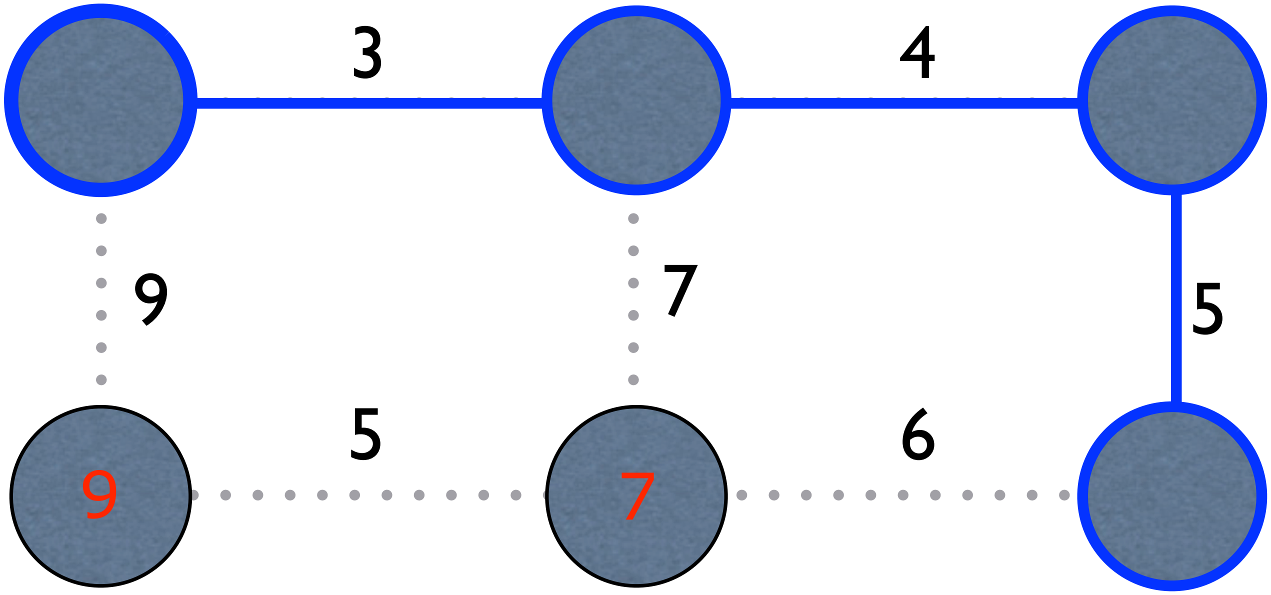

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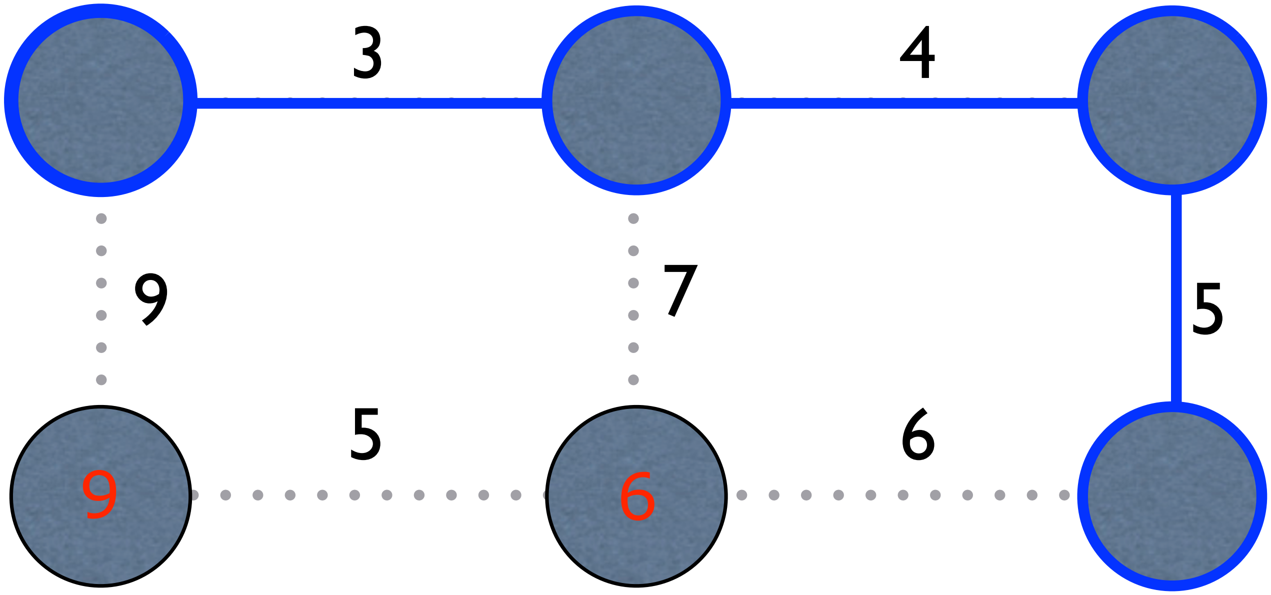

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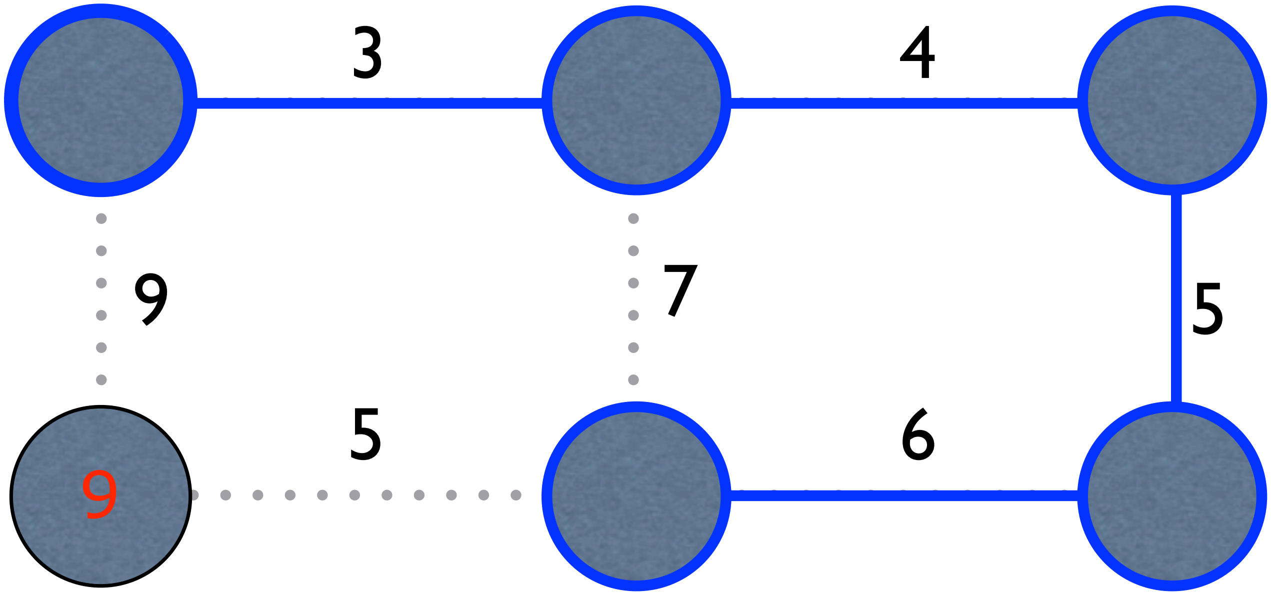

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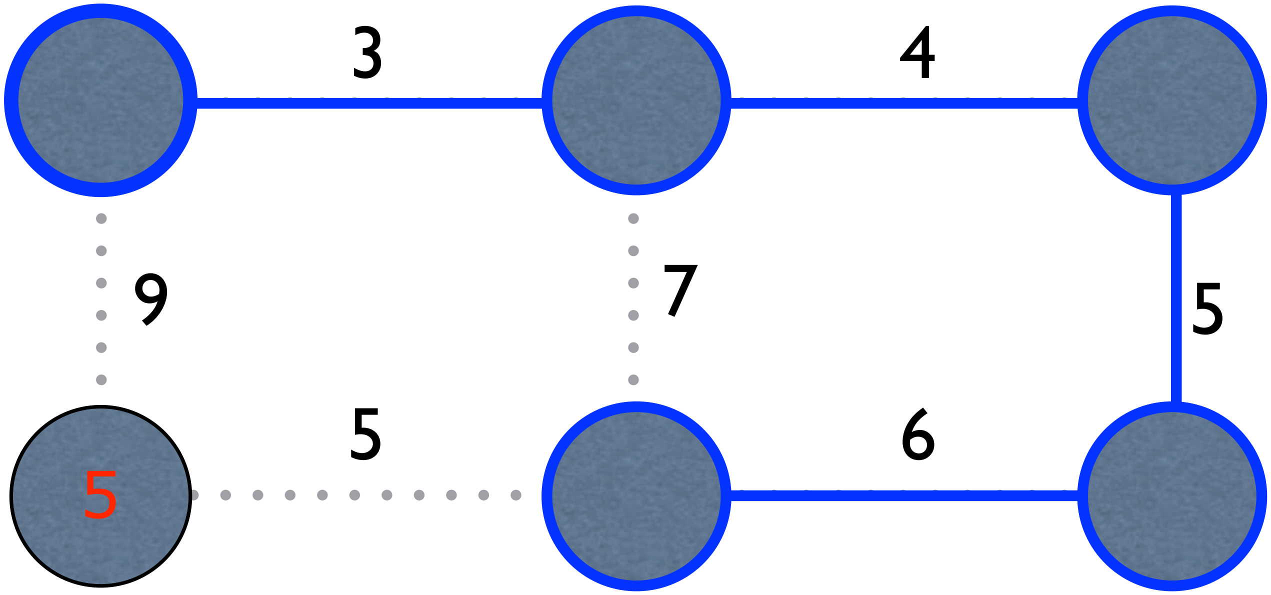

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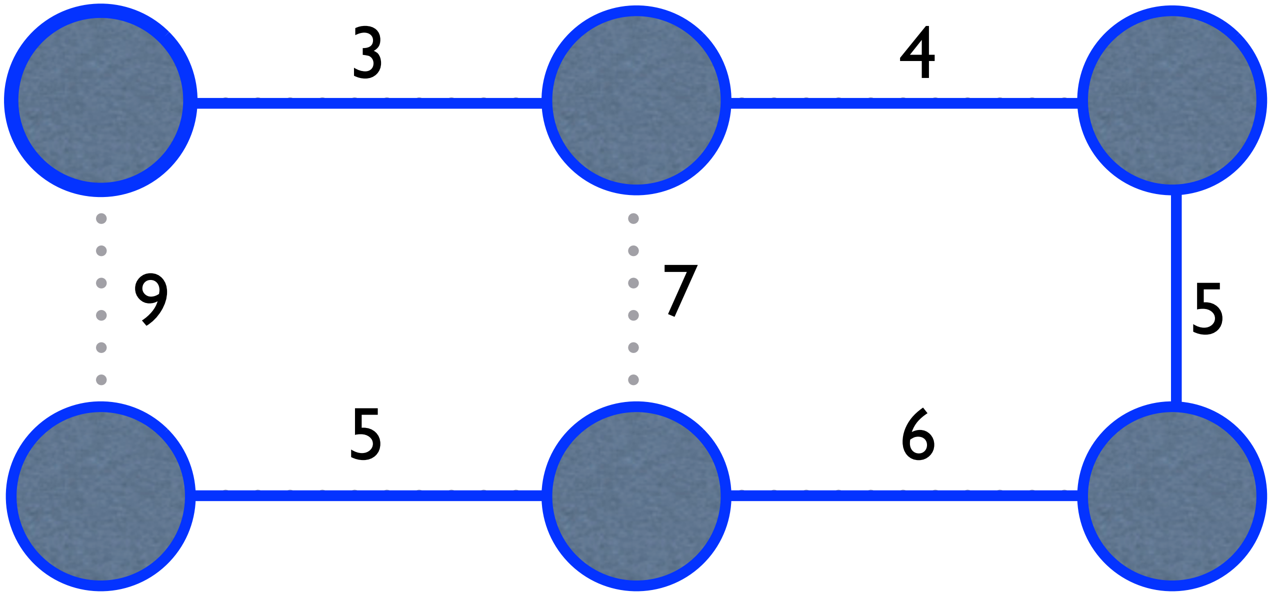


Implementation Details

- Store “previous node” like Dijkstra’s Algorithm; backtrack to construct tree after completion
- Of course, use a priority queue to keep track of edge weights. Either
  - keep track of nodes inside heap & decreaseKey
  - or just add a new copy of the node when key decreases, and call deleteMin until you see a node not in the tree

Prim’s Algorithm Justification

- At any point, we can consider the set of nodes in the tree **T** and the set outside the tree **Q**
- Whatever the MST structure of the nodes in **Q**, at least one edge must connect the MSTs of **T** and **Q**
- The greedy edge is just as good structurally as any other edge, and has minimum weight

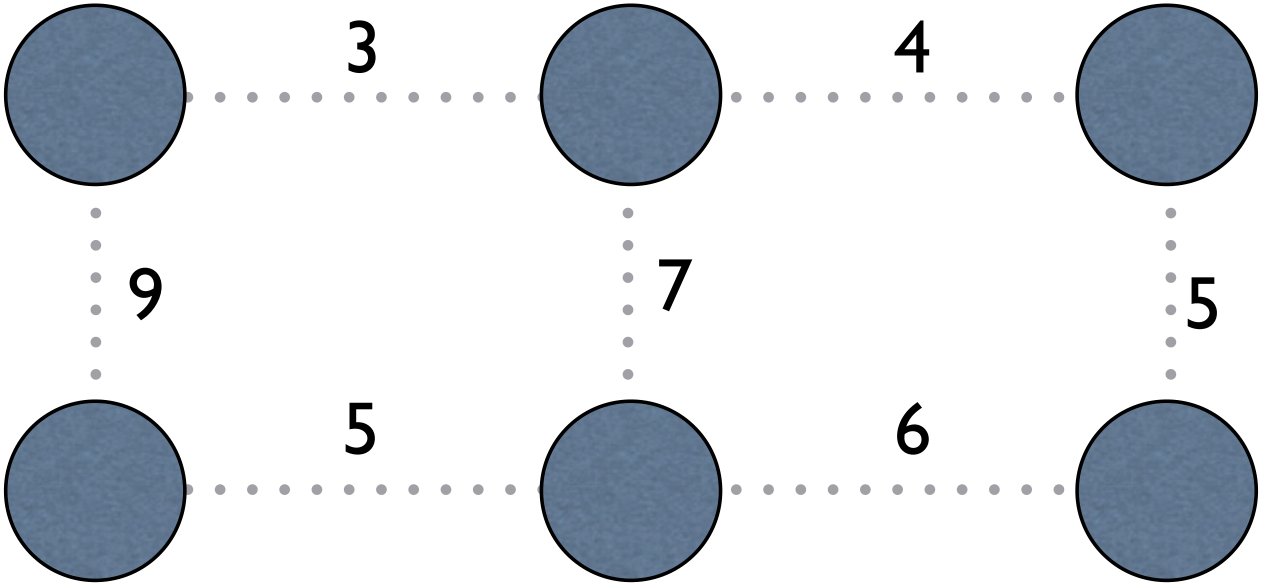
Prim’s Running Time

- Each stage requires one deleteMin O(log |V|), and there are exactly |V| stages
- We update keys for each edge, updating the key costs O(log |V|) (either an insert or a decreaseKey)
- Total time:
  O(|V| log |V| + |E| log |V|) = O(|E| log |V|)

Kruskal’s Algorithm

- Somewhat simpler conceptually, but more challenging to implement
- Algorithm: repeatedly add the shortest edge that does not cause a cycle until no such edges exist
- Each added edge performs a union on two trees; perform unions until there is only one tree
- Need special ADT for unions
  (Disjoint Set)

Kruskal’s Example



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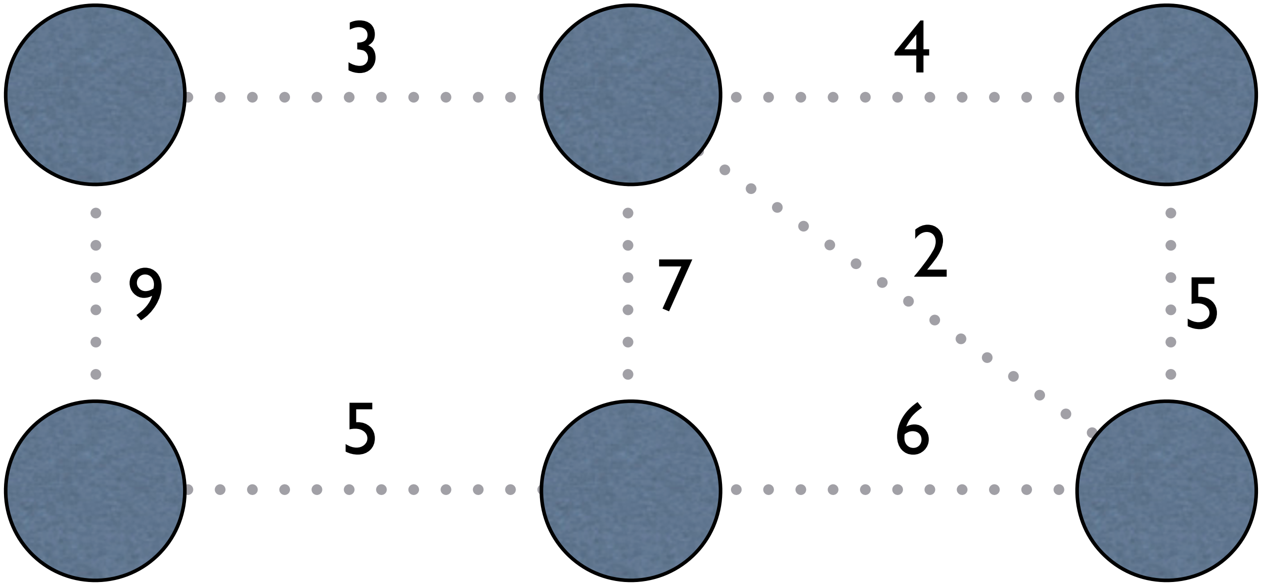

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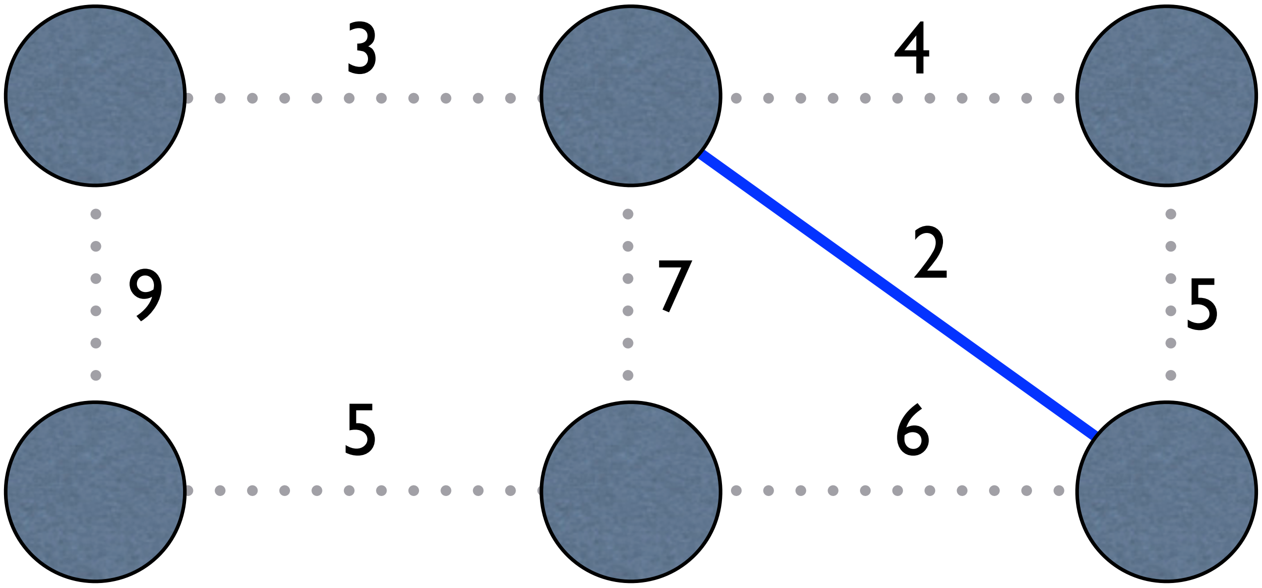
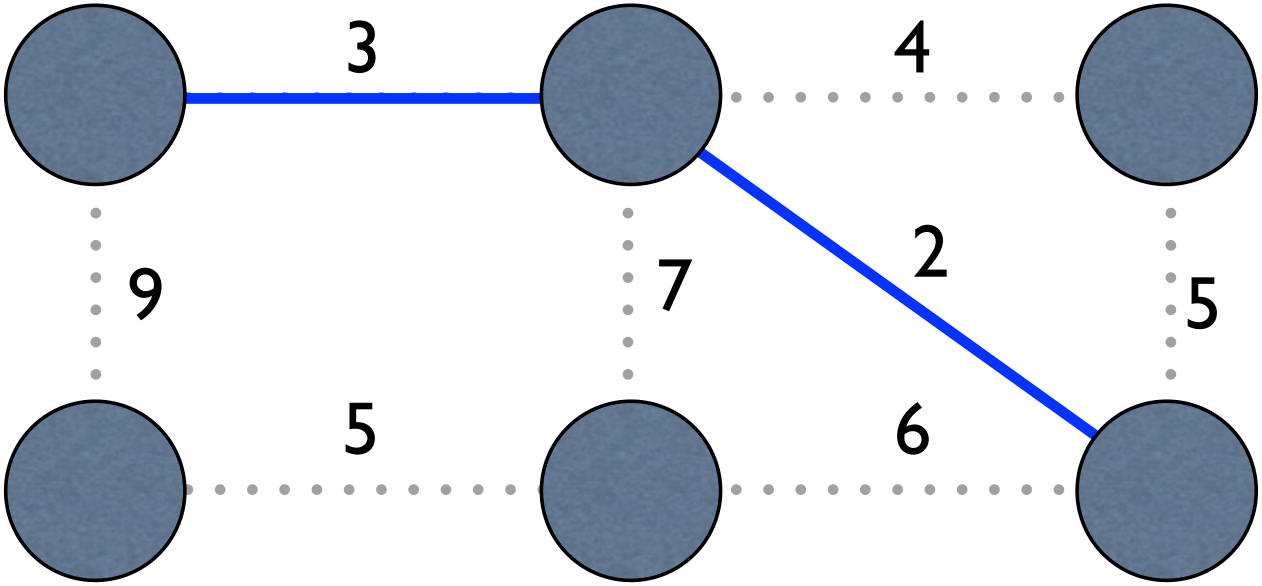
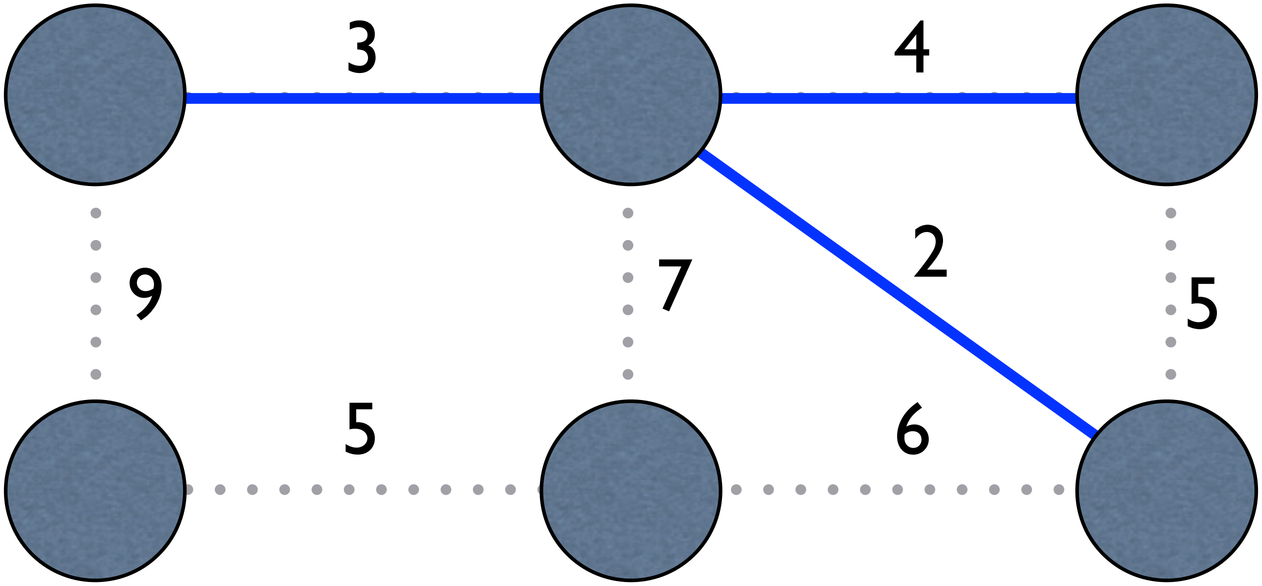


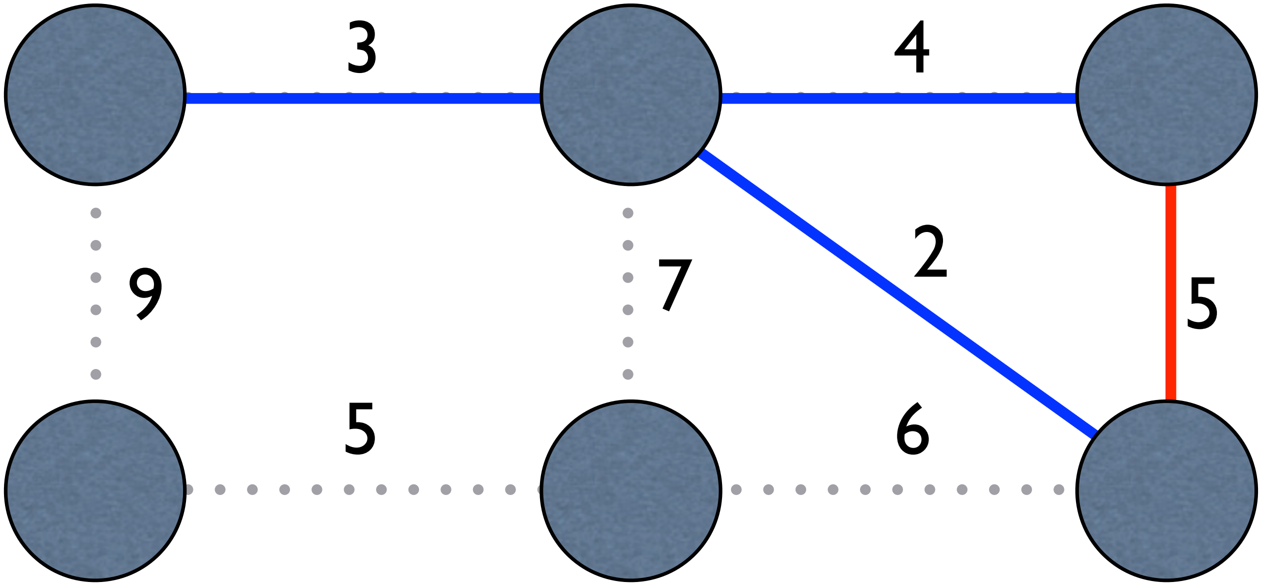
Kruskal's Example



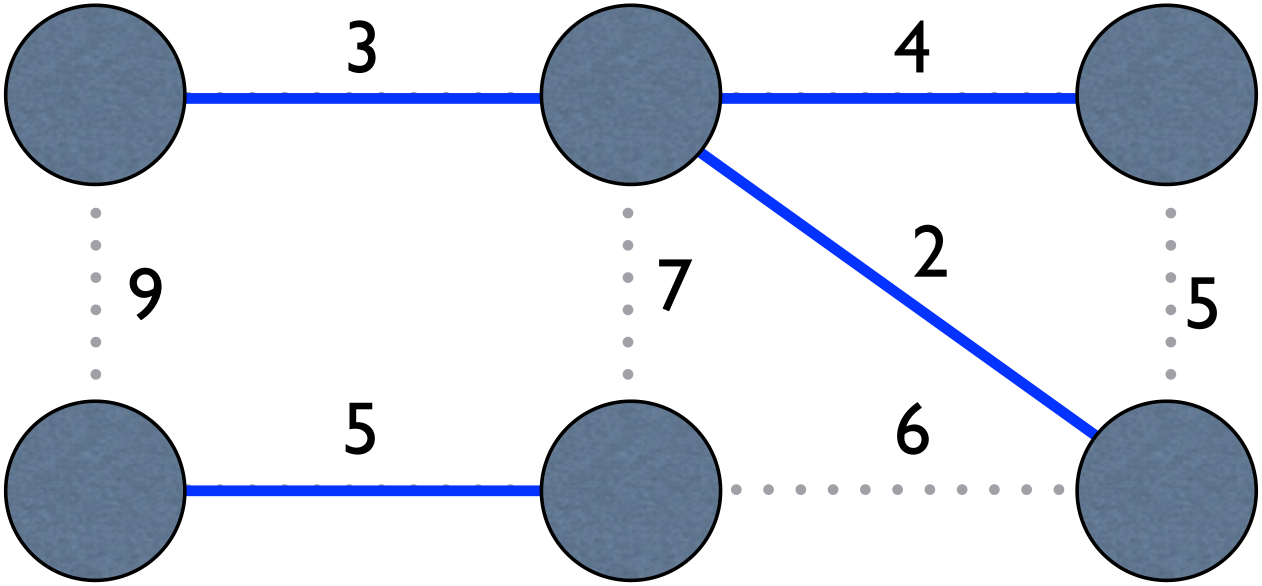
Kruskal's Example



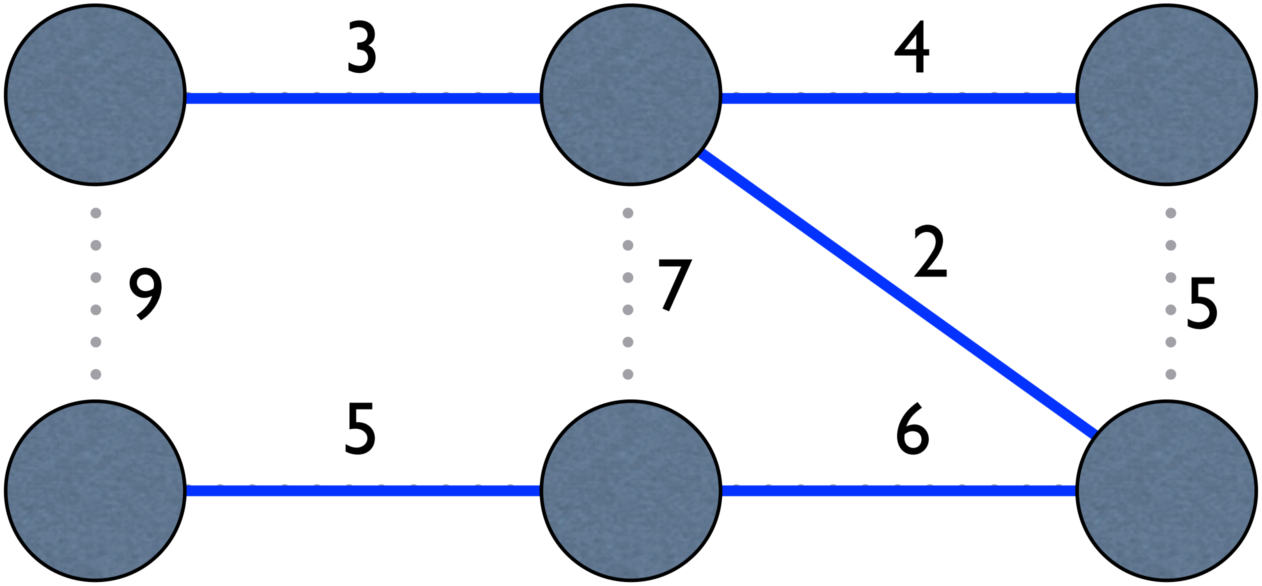
Kruskal's Example



Kruskal’s Example



Kruskal's Example



Kruskal’s Justification

- At each stage, the greedy edge **e** connects two nodes **v** and **w**
- Eventually those two nodes must be connected;
  - we must add an edge to connect trees including **v** and **w**
- We can always use **e** to connect **v** and **w**, which must have less weight since it's the greedy choice

Kruskal's Running Time

- First, buildHeap costs O(|E|)
- In the worst case, we have to call |E| deleteMins $|E| \leq |V|^2$
- Total running time O(|E| log |E|); but

$$O(|E| \log |V|^2) = O(2|E| \log |V|) = O(|E| \log |V|)$$

MST Summary

- Connect all nodes in graph using minimum weight tree
- Two greedy algorithms:
  - Prim’s: similar to Dijkstra’s. Easier to code
  - Kruskal’s: easy on paper

Disjoint Sets

Motivating Example

- One interpretation of Kruskal’s Algorithm:
  - Think of trees as sets of connected nodes
  - Merge sets by connecting nodes
  - Never merge nodes that are in the same set
- Simple idea, but how can we implement it?

Equivalence Relations

- An equivalence relation is a relation operator that observes three properties:
  - **Reflexive**: (a R a), for all a
  - **Symmetric**: (a R b) if and only if (b R a)
  - **Transitive**: (a R b) and (b R c) implies (a R c)
- Put another way, equivalence relations check if operands are in the same **equivalence class**

Equivalence Classes

- Equivalence class: the set of elements that are all related to each other via an equivalence relation
- Due to transitivity, each member can only be a member of one equivalence class
- Thus, equivalence classes are **disjoint sets**
  - Choose any distinct sets S and T, $S \cap T = \emptyset$

Disjoint Set ADT

- Collection of objects, each in an equivalence class
- **find**(x) returns the class of the object
- **union**(x,y) puts x and y in the same class
  - as well as every other relative of x and y
- Even less information than hash; no keys, no ordering

Implementation Observations

- One simple implementation would be to store the class label for each element in an array
    - O(1) lookup for **find**, O(N) for **union**
- If we store equivalent elements in linked lists, we avoid scanning the whole set during **union**
    - We can change the labels of the smaller class

Data Structure

- Store elements in equivalence (general) trees
- Use the tree’s root as equivalence class label
- **find** returns root of containing tree
- **union** merges tree
- Since all operations only search up the tree, we can store in an array

Implementation

- Index all objects from 0 to N-1
- Store a parent array such that **s[i]** is the index of **i**’s parent
- If **i** is a root, store the negative size of its tree*
- **find** follows **s[i]** until negative, returns index
- **union**(x,y) points the root of x’s tree to the root of y’s tree

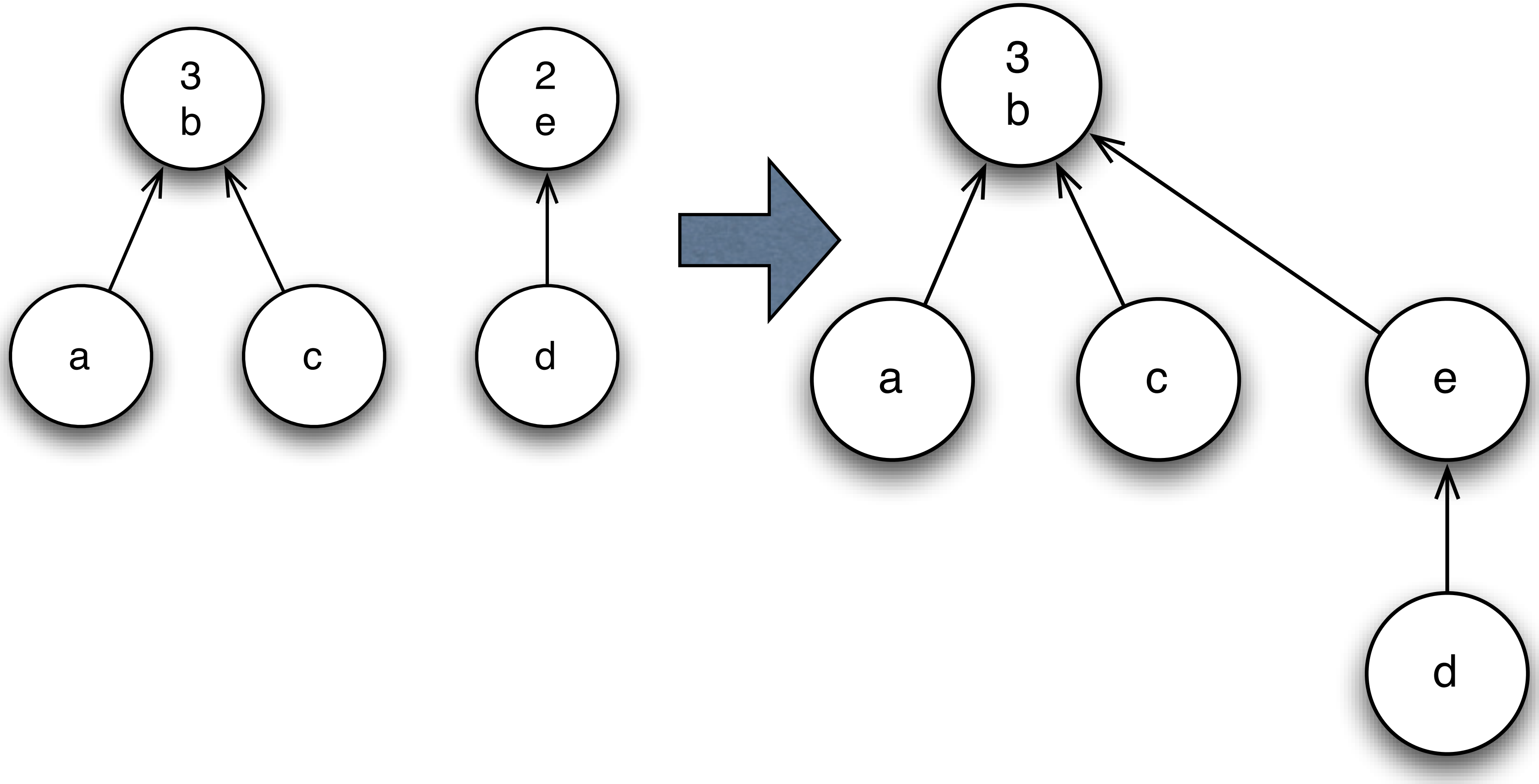
Analysis

- **find** costs the depth of the node
- **union** costs O(1) after **find**ing the roots
- Both operations depend on the height of the tree
- Since these are general trees, the trees can be arbitrarily shallow

Union by Size

- Claim: if we union by pointing the smaller tree to the larger tree’s root, the height is at most log N
- Each union increases the depths of nodes in the smaller trees
- Also puts nodes from the smaller tree into a tree at least twice the size
    - We can only double the size log N times

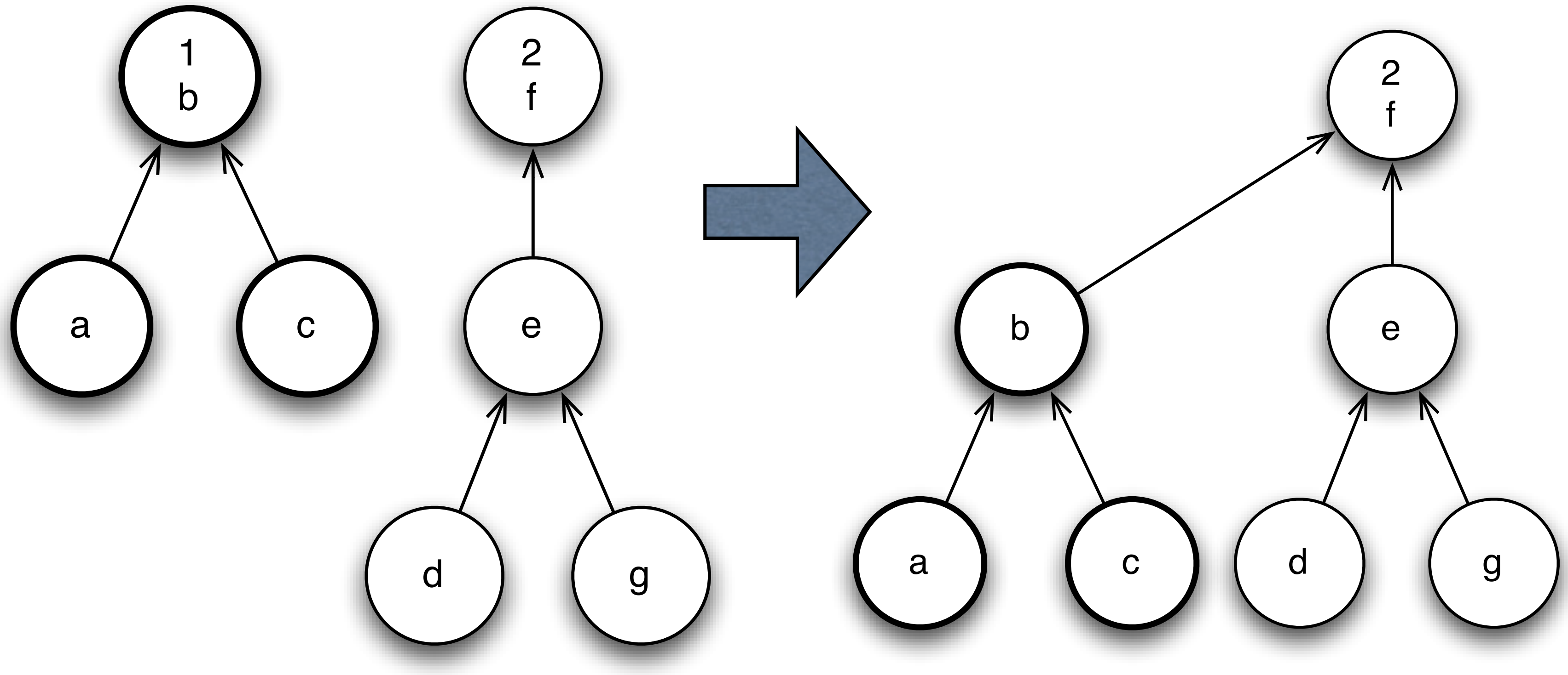
Union by Size Figure



Union by Height

- Similar method, attach the tree with less height to the taller tree
- overall height only increases if trees are equal height

Union by Height Figure



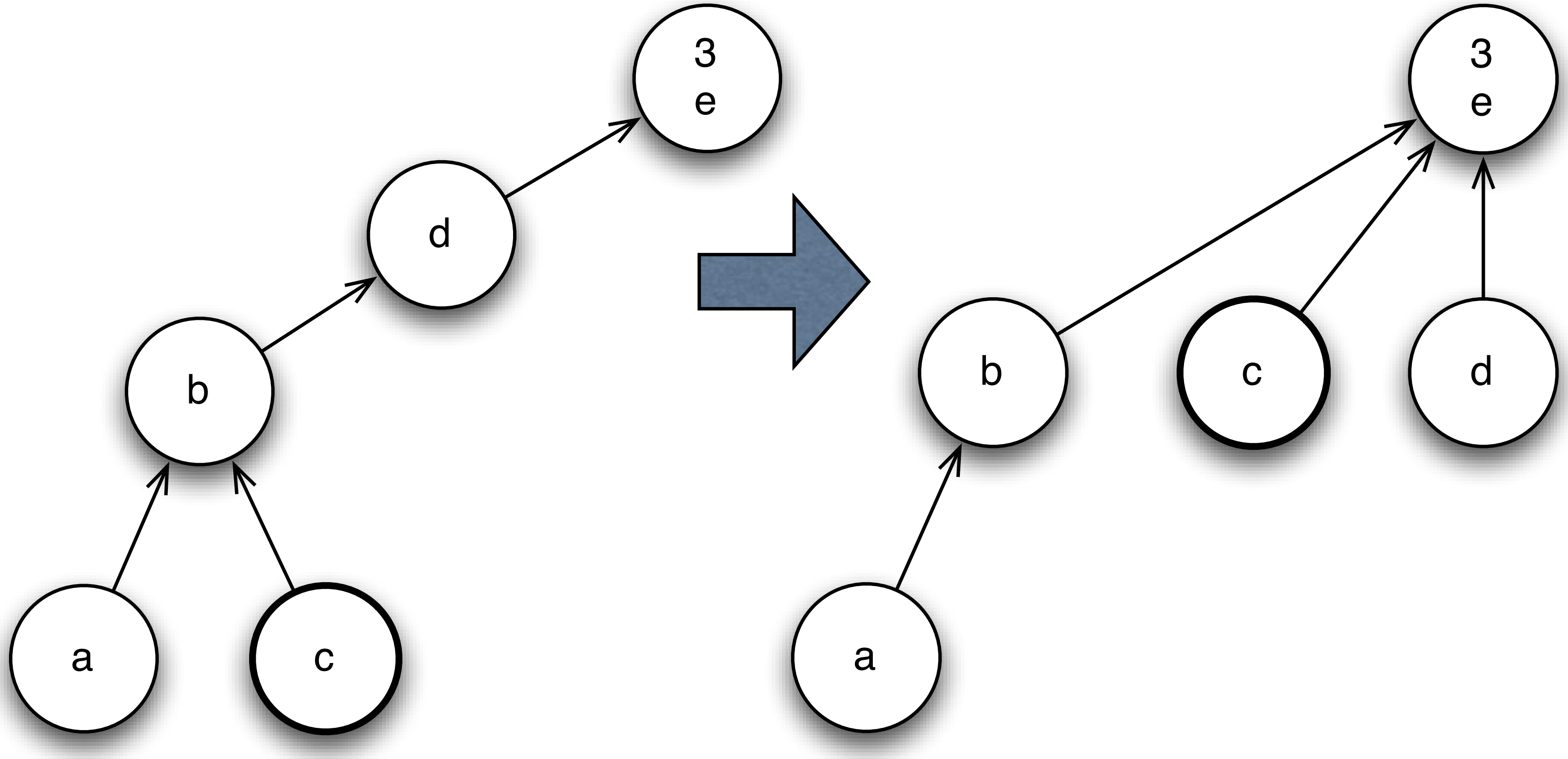
Union by Height proof

- Induction: tree of height **h** has at least 2^h nodes
- Let **T** be tree of height **h** with least nodes possible via union operations
- At last union, **T** must have had height **h-1**, because otherwise, it would have been a smaller tree of height **h**
- Since the height was updated, **T** unioned with another tree of height **h-1**, each had at least 2^{h-1} nodes resulting in at least 2^h nodes for **T**

Path Compression

- Even if we have log N tall trees, we can keep calling **find** on the deepest node repeatedly, costing O(M log N) for M operations
- Additionally, we will perform **path compression** during each **find** call
    - Point every node along the find path to root

Path Compression Figure



Union by Rank

- Path compression messes up union-by-height because we reduce the height when we compress
- We could fix the height, but this turns out to gain little, and costs **find** operations more
- Instead, rename to **union by rank**, where **rank** is just an overestimate of height
- Since heights change less often than sizes, rank/height is usually the cheaper choice

Worst Case Bound

- The algorithms described have been proven to have worst case $\Theta(M\alpha(M,N))$ where α is the inverse of Ackermann's function:

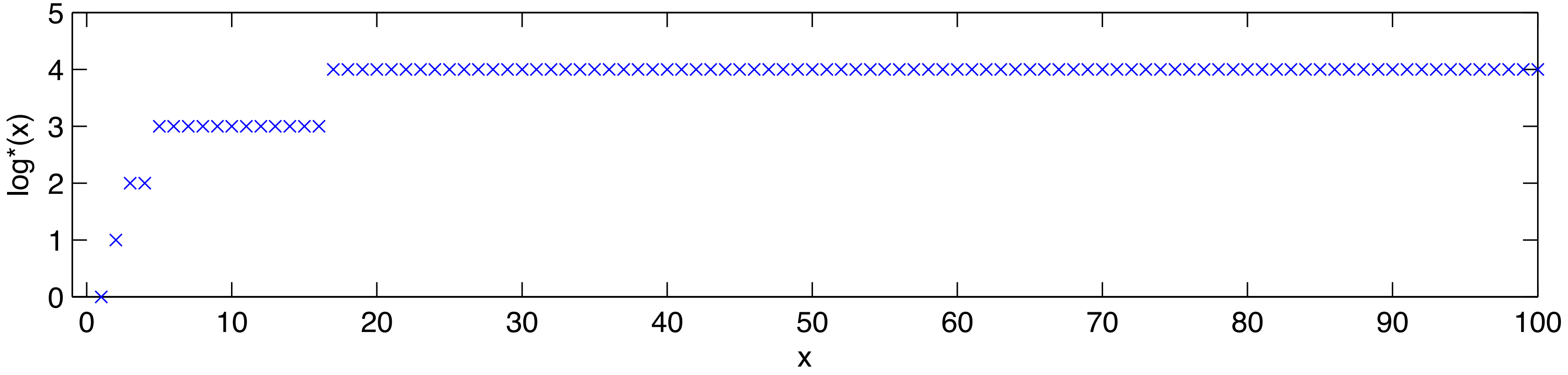
- $$\begin{aligned} A(1,j) &= 2^j \\ A(i,1) &= A(i-1,2) \\ A(i,j) &= A(i-1,A(i,j-1)) \end{aligned}$$

- $$\alpha(M,N) = \min\{i \geq 1 | A(i, \lfloor M/N \rfloor) > \log N\}$$

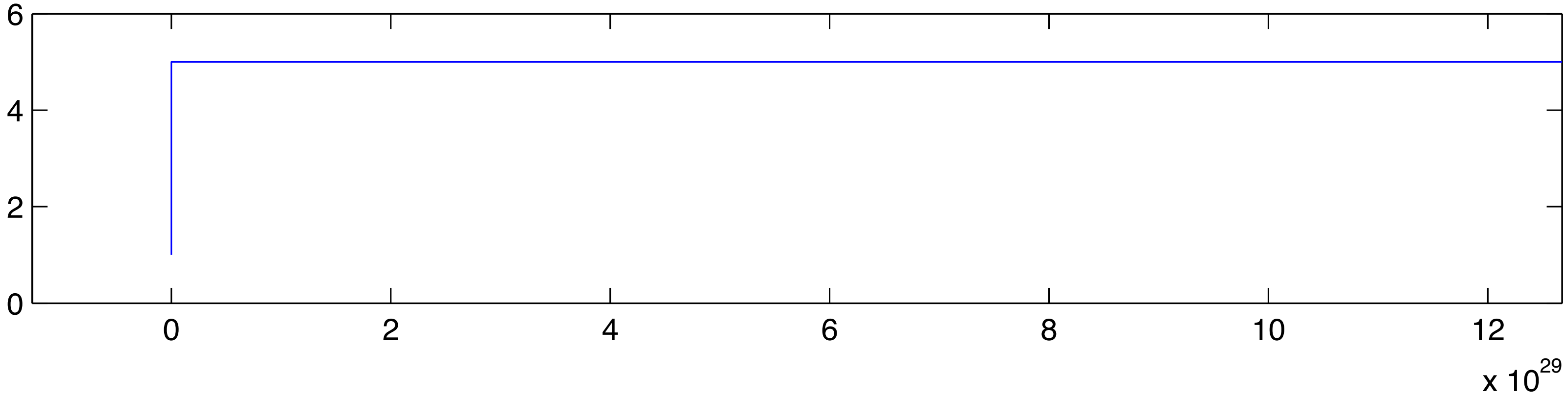
Worst Case Bound

- A slightly looser, but easier to prove/understand bound is that any sequence of $M = \Omega(N)$ operations will cost **O(M log* N)** running time
- log* N is the number of times the logarithm needs to be applied to N until the result is ≤ 1
- e.g., log*(65536) = 4 because log(log(log(log(65536)))) = 1

Log* Plots



log*(x)
5
4
3
2
1
0
0
10
20
30
40
50
60
70
80
90
100
x



6
4
2
0
0
2
4
6
8
10
12
x 10^29

Log* Steps

	N
log* N = 1	(1, 2]
log* N = 2	(2, 4]
log* N = 3	(4,16]
log* N = 4	(16, 65536]
log* N = 5	(65536, 2^{65536}]

Note about Kruskal's

- With this bound, Kruskal's algorithm needs N-1 unions, so it should cost almost linear time to perform unions
- Unfortunately the algorithm is still dominated by heap deleteMin calls, so asymptotic running time is still O(E log V)

Reading

- Weiss 9.5 (MST)
- Weiss 8.1-8.5 (Disjoint Sets)