Data Structures in Java

Session 17

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Announcements

- Homework 4 due
- Homework 5 posted
  - All-pairs shortest paths

Review

- Graphs
- Topological Sort
  - Print out a node with indegree 0,
  - update indegrees

Today's Plan

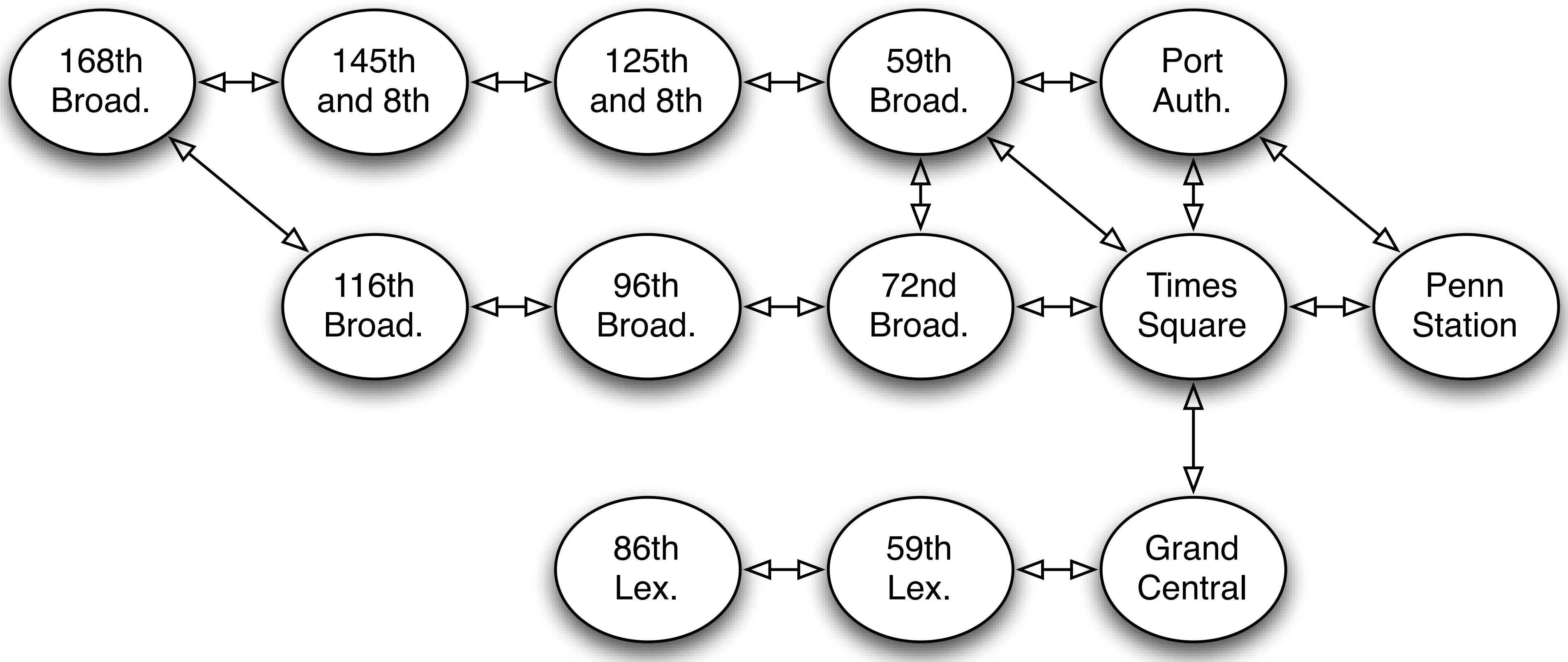
- Shortest Path algorithms
  - Breadth first search
  - Dijkstra's Algorithm
  - All-Pairs Shortest Path

Shortest Path

- Given **G = (V,E)**, and a node **s** ∈ **V**, find the shortest (weighted) path from **s** to every other vertex in **G**.
- Motivating example: subway travel
  - Nodes are junctions, transfer locations
  - Edge weights are estimated time of travel

Approximate MTA Express Stop Subgraph

- A few inaccuracies (don't use this to plan any trips)

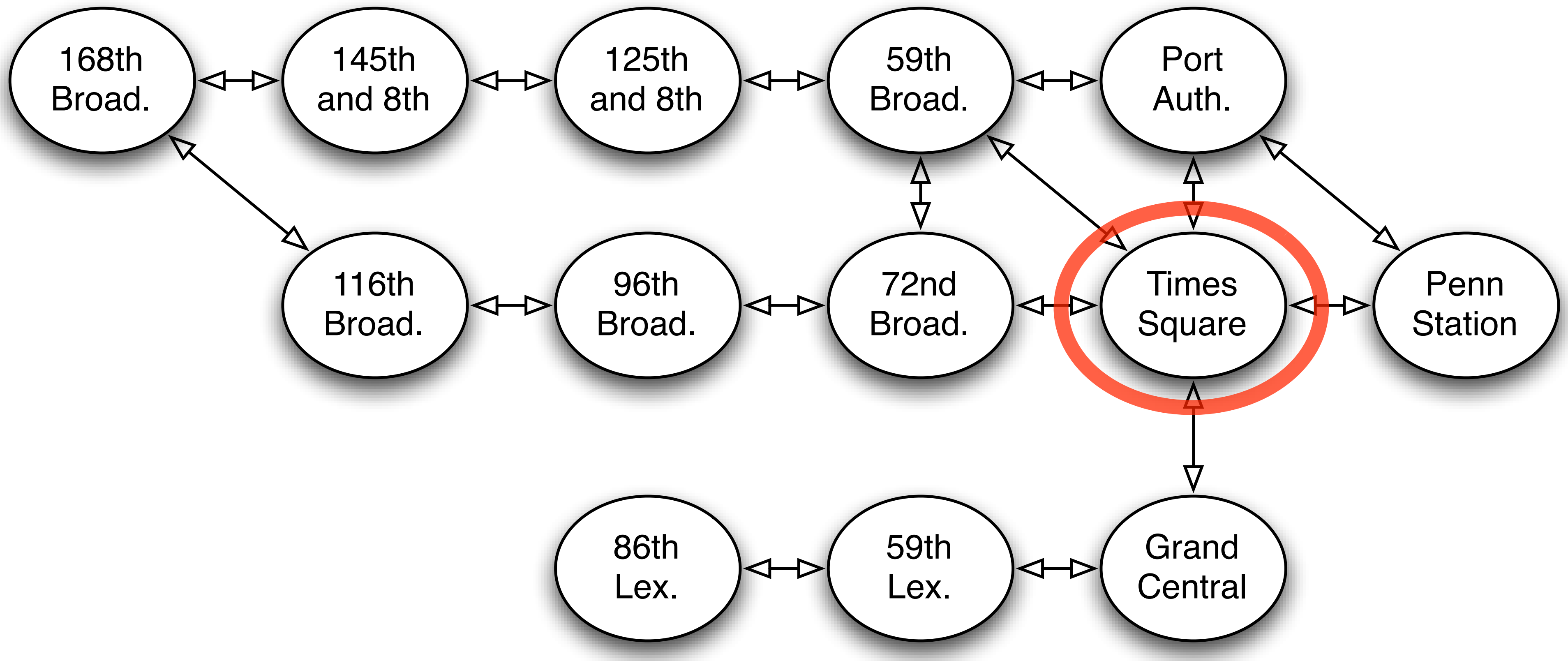


Breadth First Search

- Like a level-order traversal
- Find all adjacent nodes (level 1)
- Find *new* nodes adjacent to level 1 nodes (level 2)
- ... and so on
- We can implement this with a queue

Unweighted Shortest Path Algorithm

- Set node s’ distance to 0 and enqueue s.
- Then repeat the following:
  - Dequeue node **v**. For unset neighbor **u**:
    - set neighbor **u**’s distance to **v**’s distance +1
    - mark that we reached **v** from **u**
    - enqueue **u**



	168th Broad.	145th Broad.	125th 8th	59th Broad.	Port Auth.	116th Broad.	96th Broad.	72nd Broad.	Times Sq.	Penn St.	86th Lex.	59th Lex.	Grand Centr.
dist									0				
prev									source				

Weighted Shortest Path

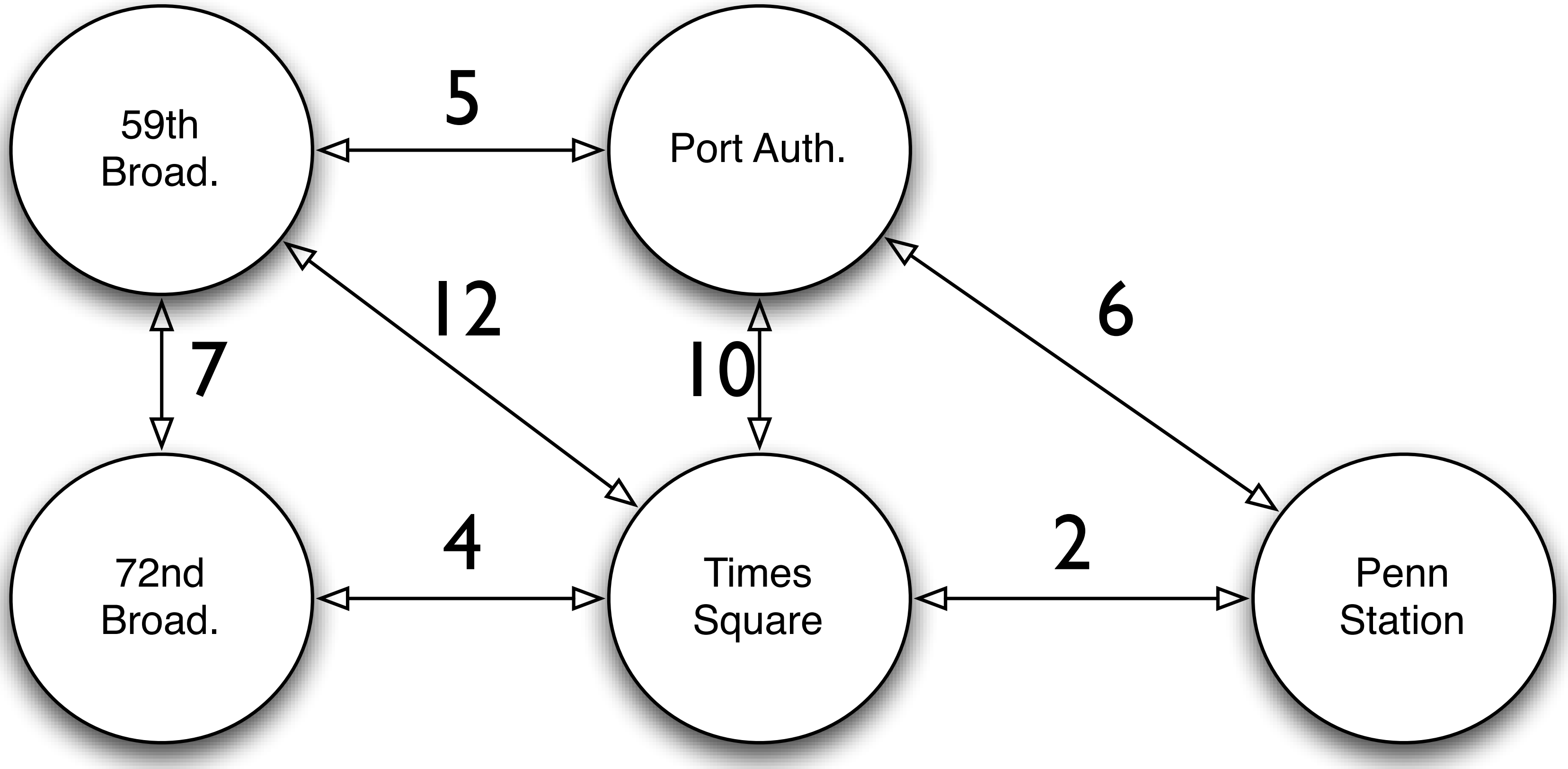
- The problem becomes more difficult when edges have different weights
- Weights represent different costs on using that edge
- Standard algorithm is **Dijkstra's Algorithm**

Dijkstra's Algorithm

- Keep distance overestimates **D(v)** for each node **v** (all non-source nodes are initially infinite)
- 1. Choose node **v** with smallest *unknown* distance
- 2. Declare that **v**'s shortest distance is *known*
- 3. Update distance estimates for neighbors

Updating Distances

- For each of **v**'s neighbors, **w**,
- if min(**D(v)+ weight(v,w)**, **D(w)**)
  - i.e., update **D(w)** if the path going through **v** is cheaper than the best path so far to **w**



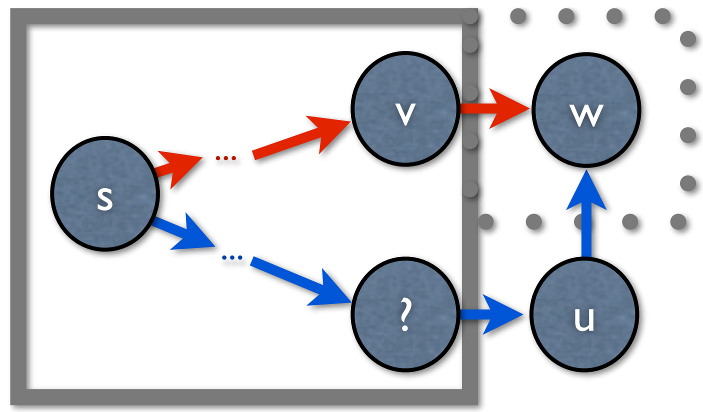
59th Broad.	Port Auth.	72nd Broad	Times Sq.	Penn St.
inf	inf	inf	inf	0
?	?	?	?	home

Dijkstra’s Algorithm Analysis

- First, convince ourselves that the algorithm works.
- At each stage, we have a set of nodes whose shortest paths we know
- In the base case, the set is the source node.
- Inductive step: if we have a correct set, is greedily adding the shortest neighbor correct?

Proof by Contradiction (Sketch)

- Contradiction: Dijkstra's finds a shortest path to node **w** through **v**, but there exists an even shorter path
- This shorter path must pass from inside our known set to outside.
- Call the 1st node in cheaper path outside our set **u**



- The path to **u** must be shorter than the path to **w**
  - But then we would have chosen **u** instead

Computational Cost

- If the graph is dense, we scan the vertices to find the minimum edge **O(V)**
- This happens **|V|** times
- We also update the distances once per edge, **O(|E|)**
- Thus, total running time is $O(|E| + |V|^2)$

Computational Cost (sparse)

- Keep a priority queue of all unknown nodes
- Each stage requires a **deleteMin**, and then some **decreaseKey**s (the # of neighbors of node)
- We call **decreaseKey** once per edge, we call **deleteMin** once per vertex
- Both operations are $O(\log |V|)$
- Total cost: $O(|E| \log |V| + |V| \log |V|) = O(|E| \log |V|)$

All Pairs Shortest Path

- Dijkstra's Algorithm finds shortest paths from one node to all other nodes
- What about computing shortest paths for all pairs of nodes?
- We can run Dijkstra's |V| times. Total cost: $O(|V|^3)$
- Floyd-Warshall algorithm is often faster in practice (though same asymptotic time)

Recursive Motivation

- Consider the set of numbered nodes **1** through **k**
- The shortest path between any node **i** and **j** using only nodes in the set {**1**, ..., **k**} is the minimum of
  - shortest path from **i** to **j** using nodes {**1**, ..., **k-1**}
  - shortest path from **i** to **j** using node **k**
- dist(i,j,k) = min(dist(i,j,k-1),
  dist(i,k,k-1)+dist(k,j,k-1))

Dynamic Programming

- Instead of repeatedly computing recursive calls, store lookup table
- To compute dist(i,j,k) for any i,j, we only need to look up dist(-,-, k-1)
  - but never k-2, k-3, etc.
- We can incrementally compute the path matrix for k=0, then use it to compute for k=1, then k=2...

Floyd-Warshall Code

- Initialize `d = ` weight matrix
-
```
for (k=0; k<N; k++)
  for (i=0; i<N; i++)
    for (j=0; j<N; j++)
      if (d[i][j] > d[i][k]+d[k][j])
        d[i][j] = d[i][k] + d[k][j];
```
- Additionally, we can store the actual path by keeping a “midpoint” matrix

Midpoint Matrix

- We can store the N^2 paths efficiently with a midpoint matrix:

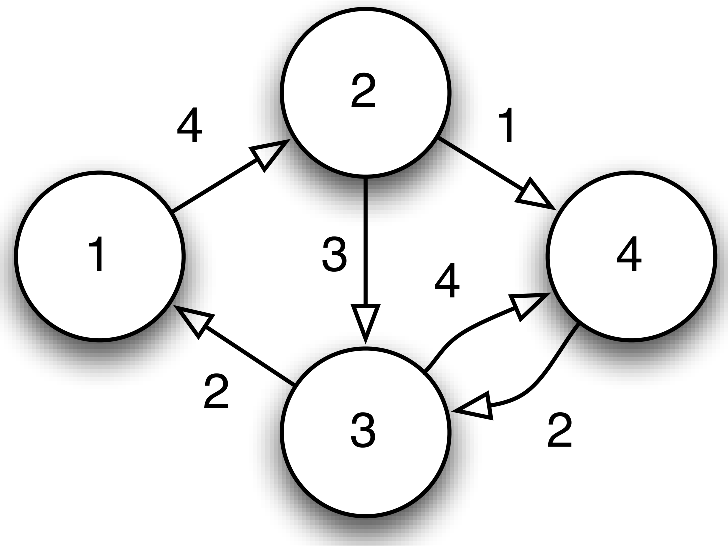
  path(i,j) = path(i, midpoint[i][j]) +
  path(midpoint[i][j], j)

- We only need a NxN matrix to store all the paths

All Pairs Shortest Path Example

k=0

	1	2	3	4
1	-	4	-	-
2	-	-	3	1
3	2	-	-	4
4	-	-	2	-



Transitive Closure

- For any nodes i, j, is there a path from i to j?
- Instead of computing shortest paths, just compute Boolean if a path exists
- path(i,j,k) = path(i,j,k-1) OR
  path(i,k,k-1) AND path(k,j,k-1)
- Transitive closure can tell you whether a graph is **connected**

Reading

- Weiss Section 9.1-9.3,
- Weiss Section 10.3.4
  (All-Pairs Shortest Path)