Data Structures in Java

Session 17

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Announcements

- Homework 4 due
- Homework 5 posted
  - All-pairs shortest paths

Review

- Graphs
- Topological Sort
  - Print out a node with indegree 0,
  - update indegrees

Today's Plan

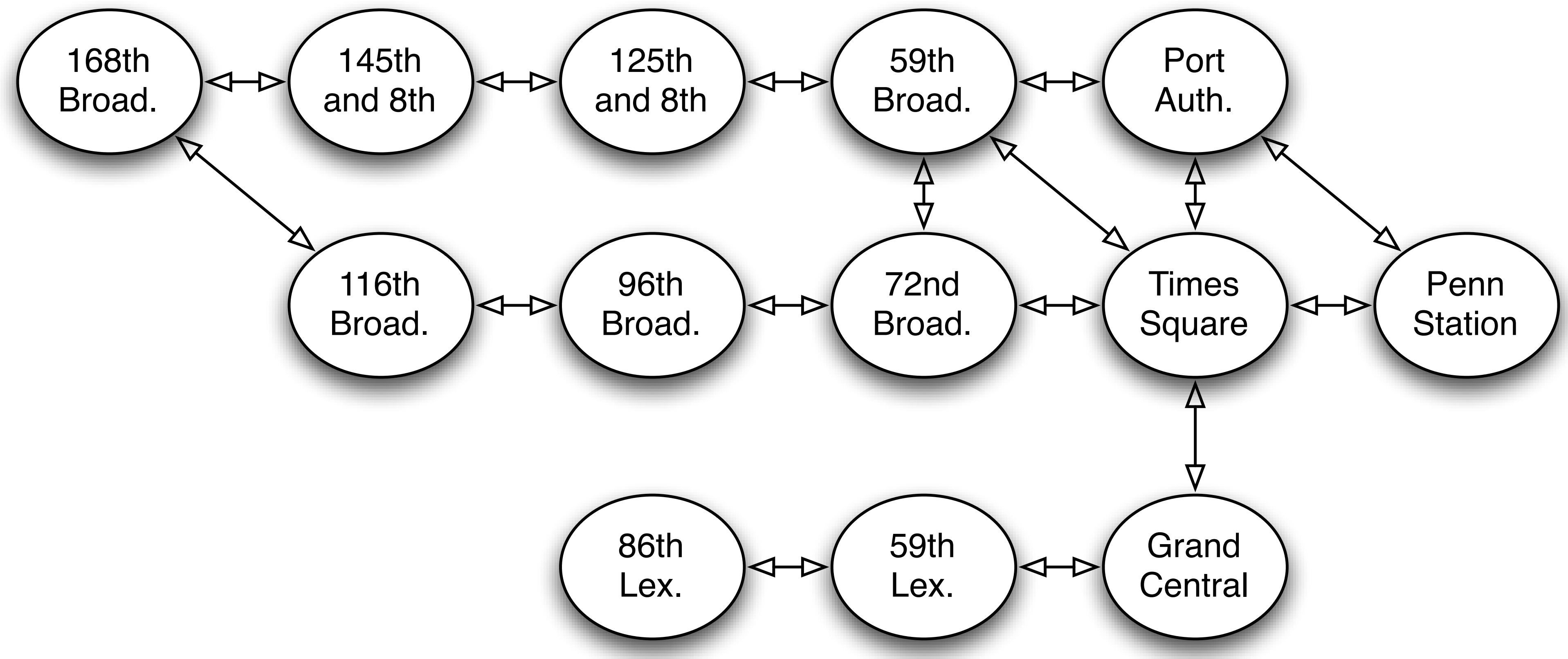
- Shortest Path algorithms
  - Breadth first search
  - Dijkstra's Algorithm
  - All-Pairs Shortest Path

Shortest Path

- Given **G = (V,E)**, and a node **s** ∈ **V**, find the shortest (weighted) path from **s** to every other vertex in **G**.
- Motivating example: subway travel
  - Nodes are junctions, transfer locations
  - Edge weights are estimated time of travel

Approximate MTA Express Stop Subgraph

- A few inaccuracies (don’t use this to plan any trips)

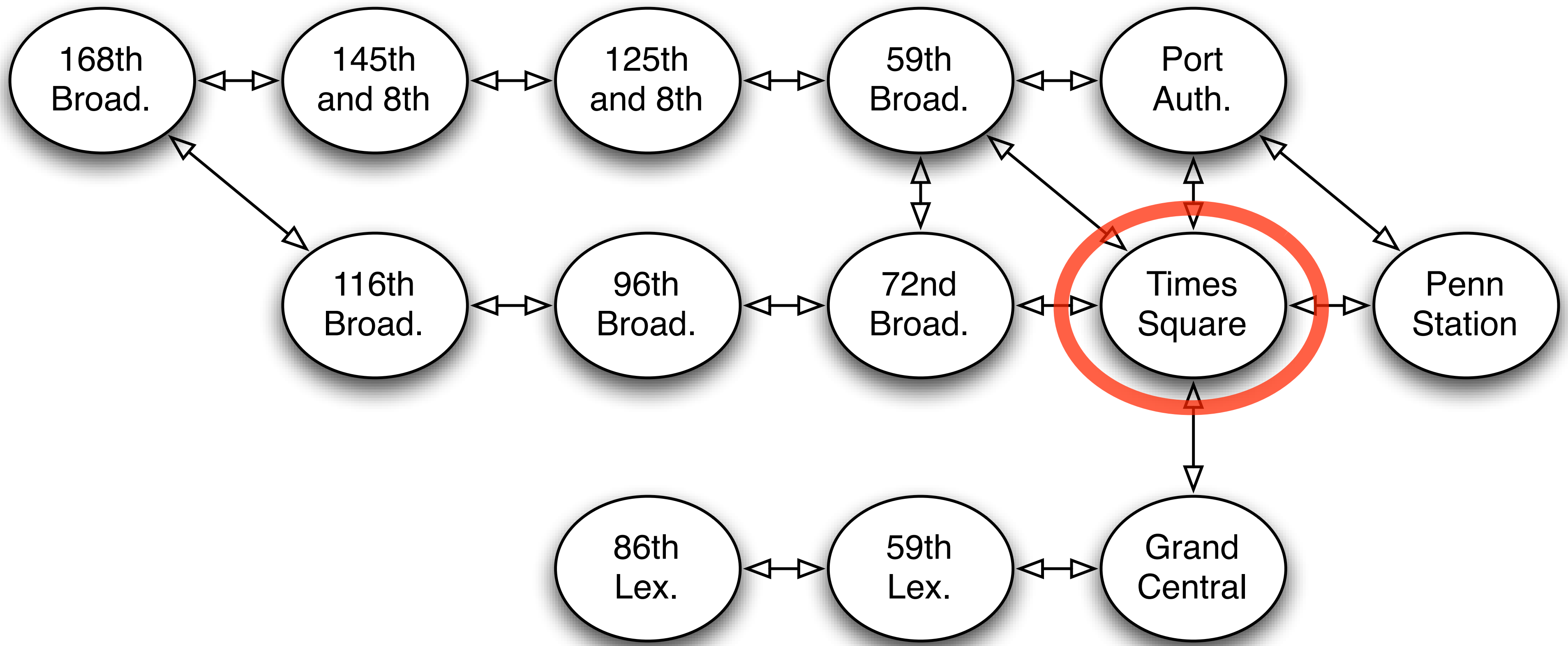


Breadth First Search

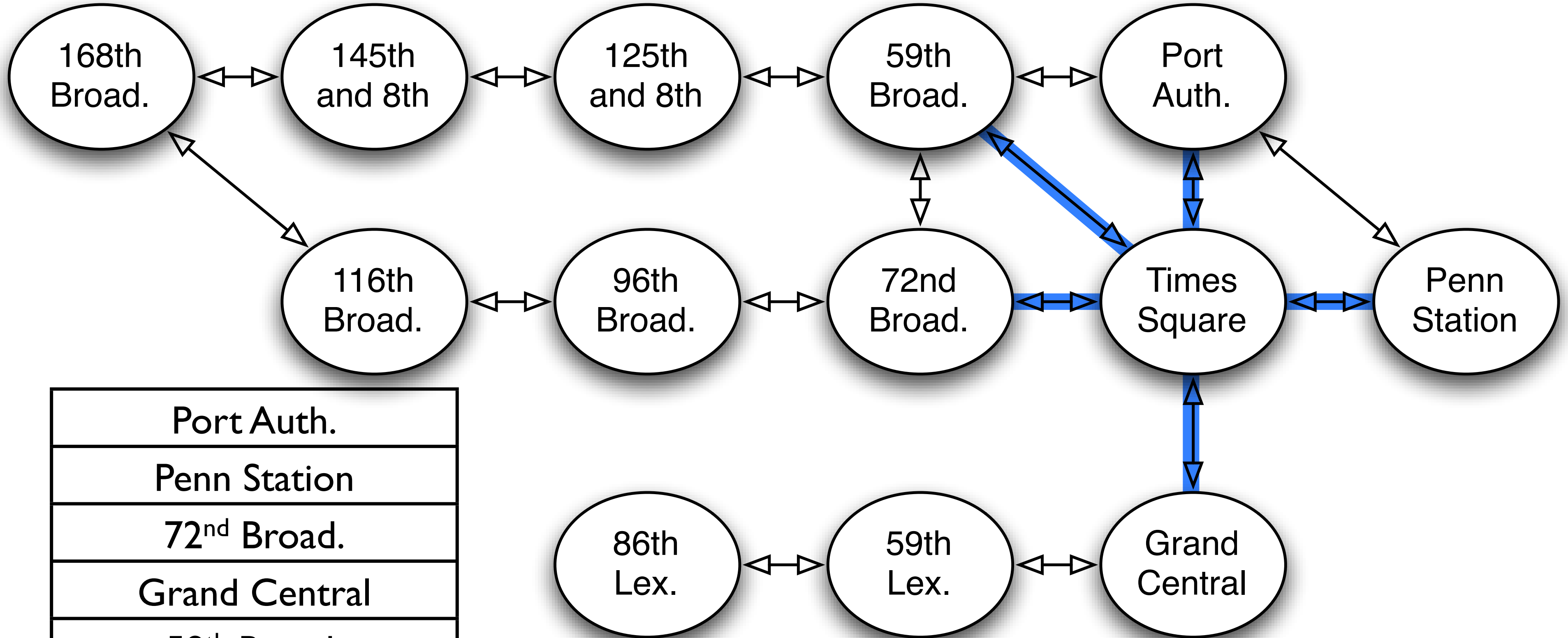
- Like a level-order traversal
- Find all adjacent nodes (level 1)
- Find *new* nodes adjacent to level 1 nodes (level 2)
- ... and so on
- We can implement this with a queue

Unweighted Shortest Path Algorithm

- Set node s' distance to 0 and enqueue s.
- Then repeat the following:
  - Dequeue node **v**. For unset neighbor **u**:
    - set neighbor **u**'s distance to **v**'s distance +1
    - mark that we reached **v** from **u**
    - enqueue **u**

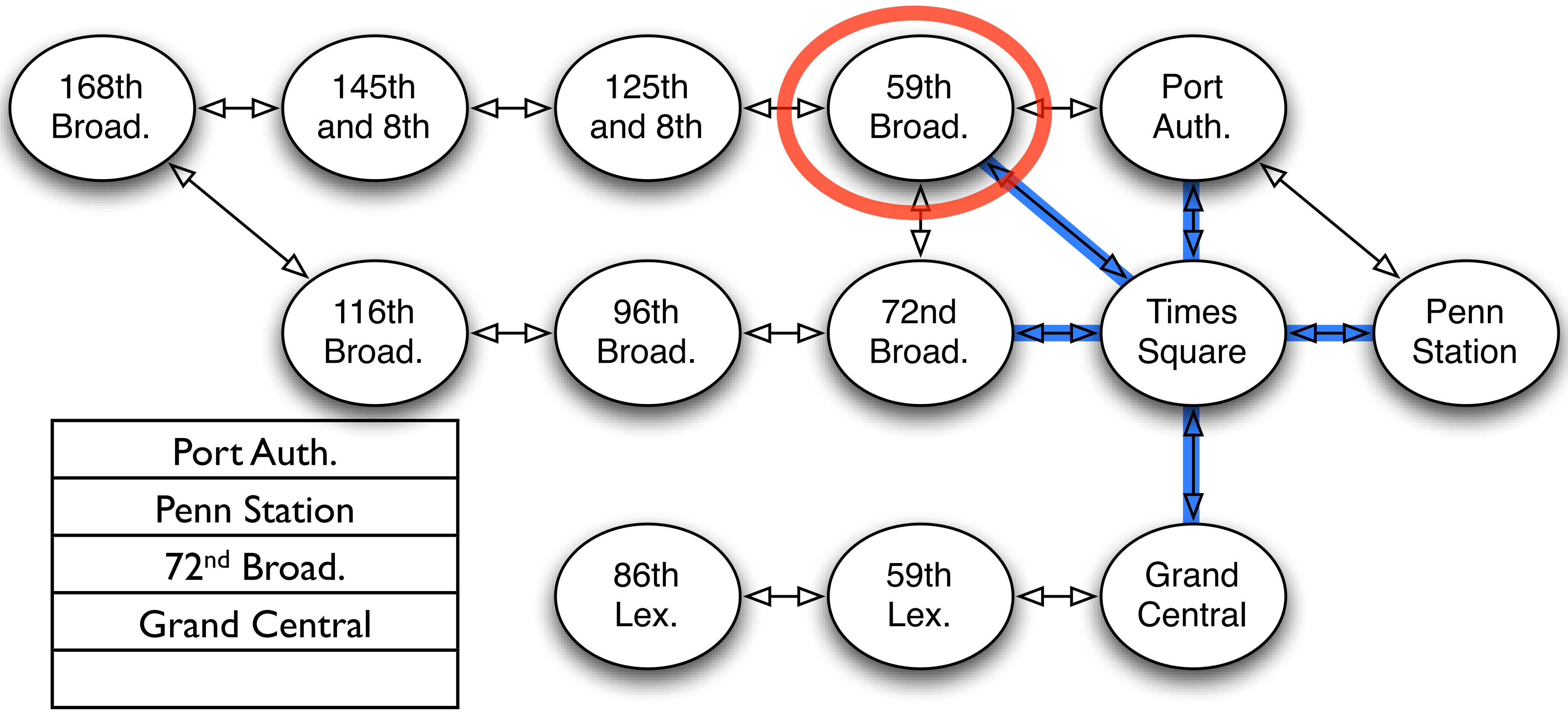


	168th Broad.	145th Broad.	125th 8th	59th Broad.	Port Auth.	116th Broad.	96th Broad.	72nd Broad.	Times Sq.	Penn St.	86th Lex.	59th Lex.	Grand Centr.
dist									0				
prev									source				



Port Auth.
Penn Station
72nd Broad.
Grand Central
59th Broad.

	168th Broad.	145th Broad.	125th 8th	59th Broad.	Port Auth.	116th Broad.	96th Broad.	72nd Broad.	Times Sq.	Penn St.	86th Lex.	59th Lex.	Grand Centr.
dist				1	1			1	0	1			1
prev				Times Sq.	Times Sq.			Times Sq.	source	Times Sq.			Times Sq.



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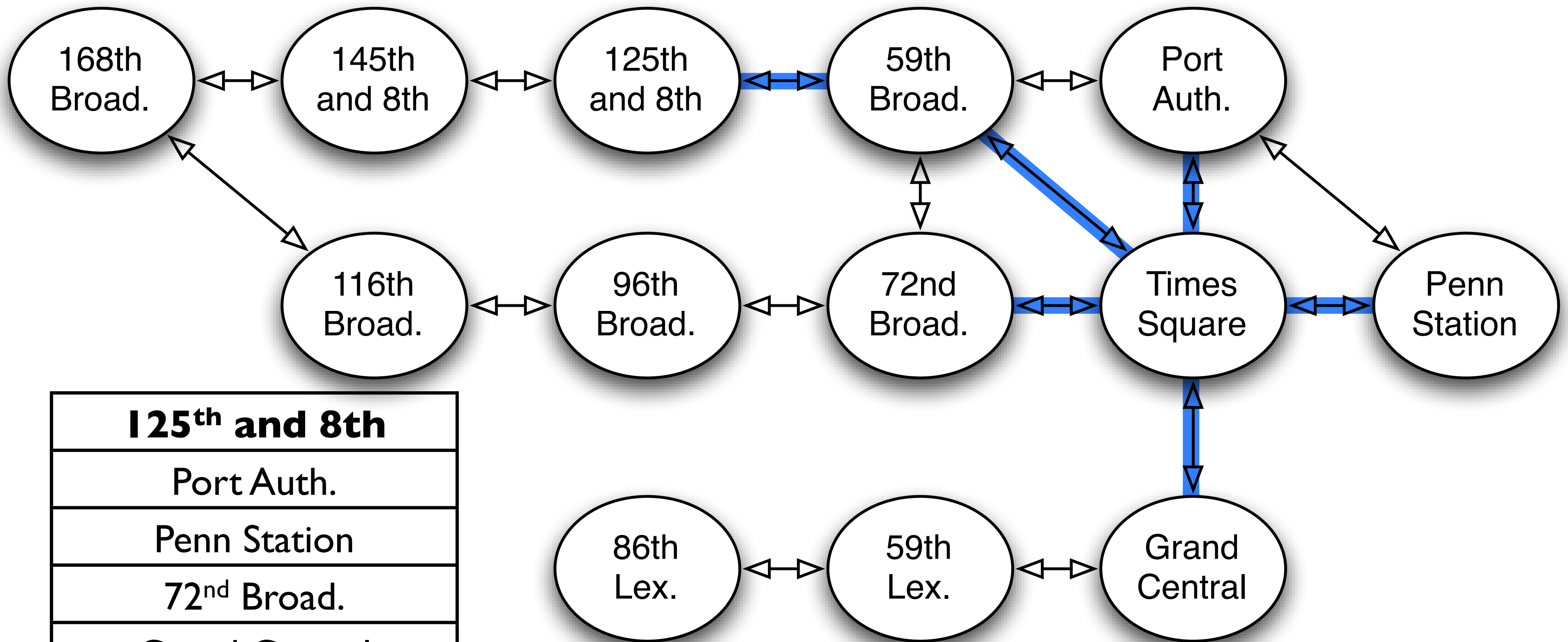
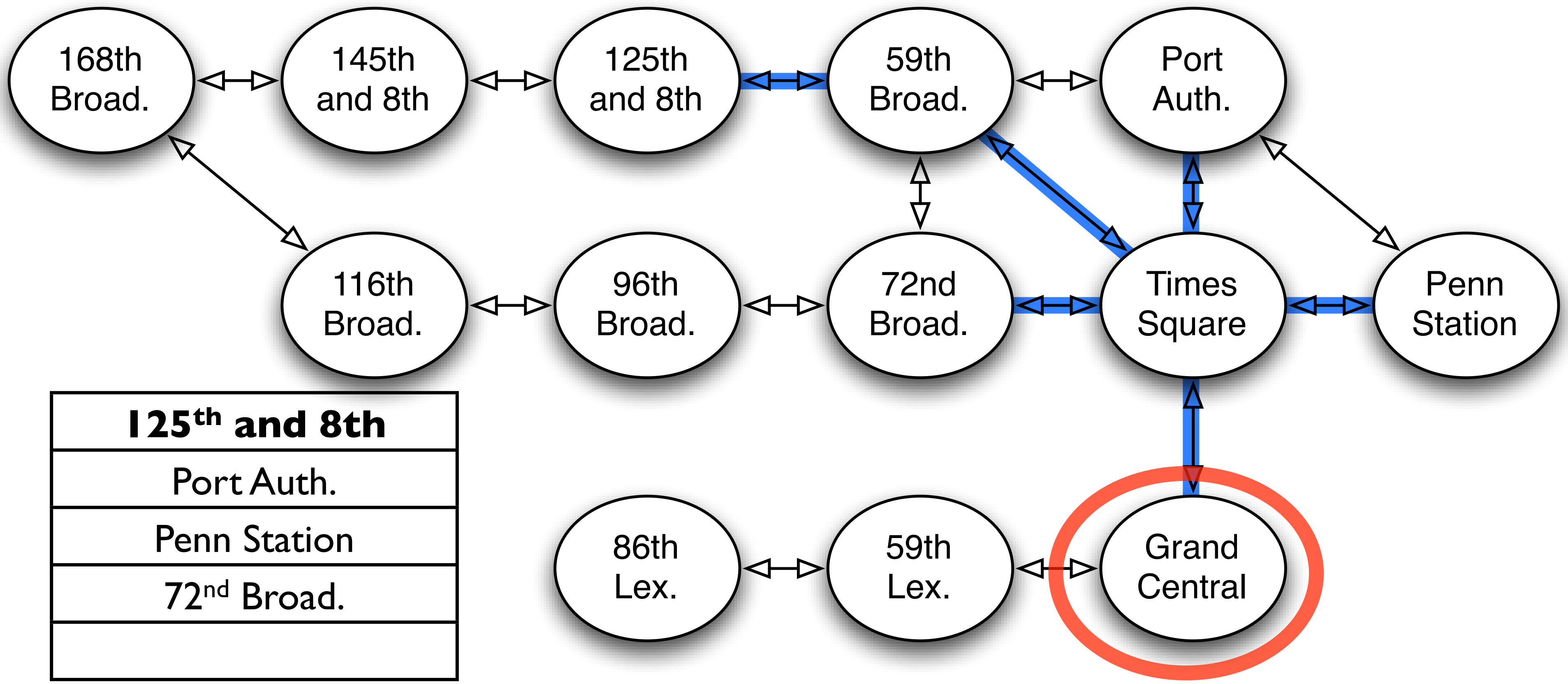
dist				1	1			1	0	1			1
prev				Times Sq.	Times Sq.			Times Sq.	source	Times Sq.			Times Sq.



125th and 8th
Port Auth.
Penn Station
72nd Broad.
Grand Central

	168th Broad.	145th Broad.	125th 8th	59th Broad.	Port Auth.	116th Broad.	96th Broad.	72nd Broad.	Times Sq.	Penn St.	86th Lex.	59th Lex.	Grand Centr.
dist			2	1	1			1	0	1			1
prev			59th Broad.	Times Sq.	Times Sq.			Times Sq.	source	Times Sq.			Times Sq.



125th and 8th
Port Auth.
Penn Station
72nd Broad.

	168th Broad.	145th Broad.	125th 8th	59th Broad.	Port Auth.	116th Broad.	96th Broad.	72nd Broad.	Times Sq.	Penn St.	86th Lex.	59th Lex.	Grand Centr.
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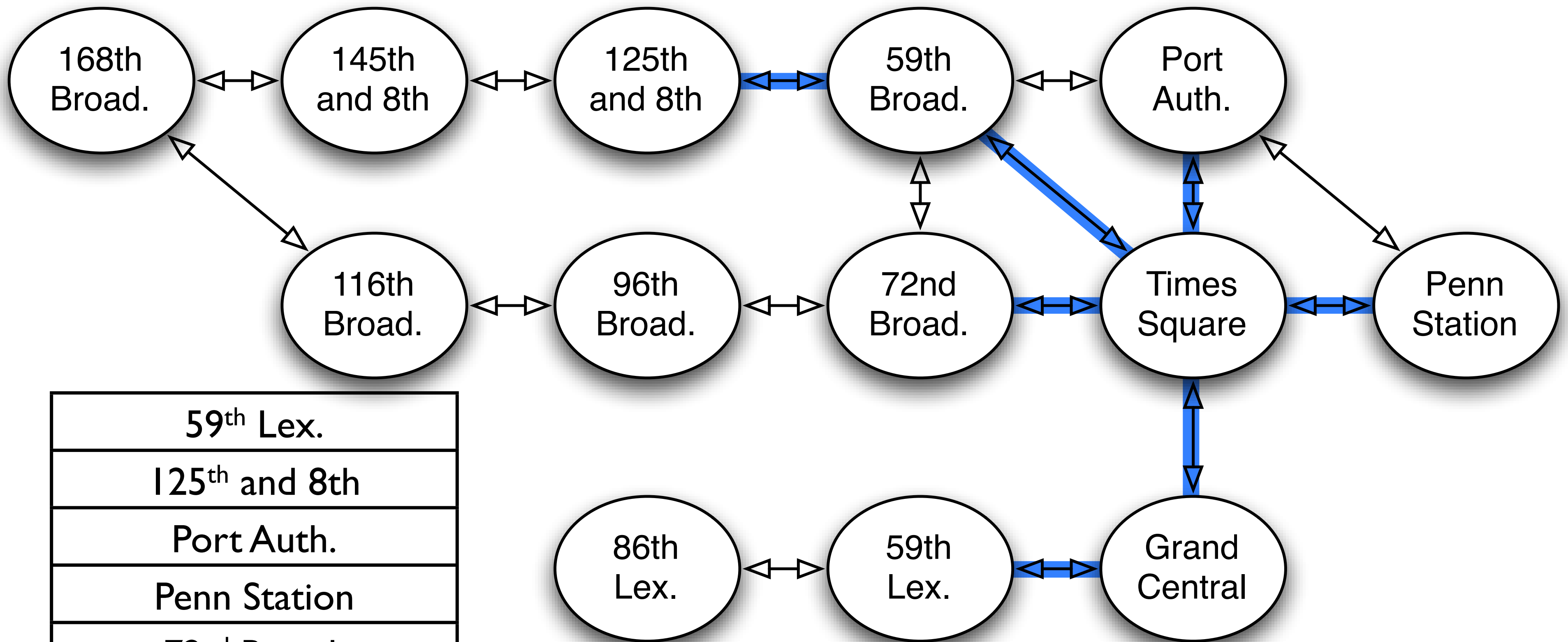
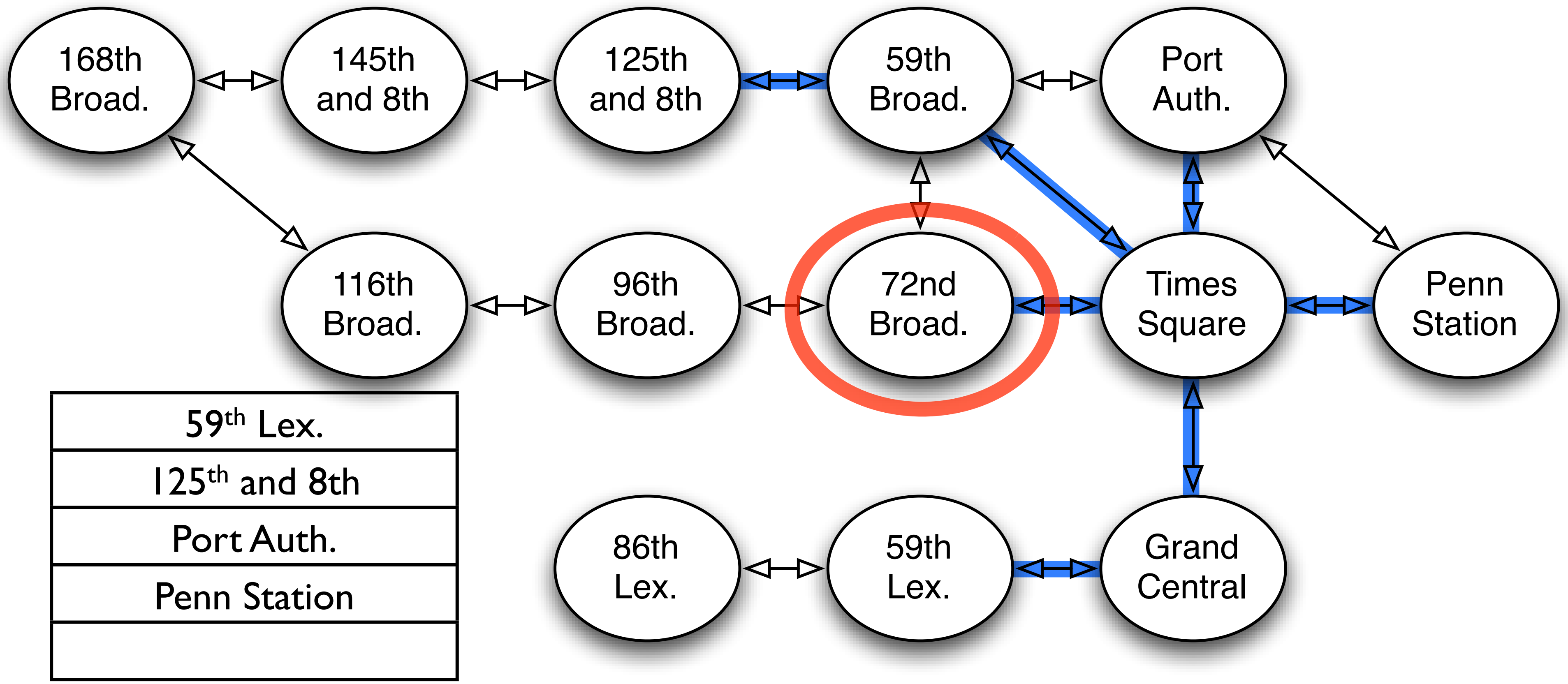
prev			59th Broad.	Times Sq.	Times Sq.			Times Sq.	source	Times Sq.			Times Sq.



59th Lex.
125th and 8th
Port Auth.
Penn Station
72nd Broad.

	168th Broad.	145th Broad.	125th 8th	59th Broad.	Port Auth.	116th Broad.	96th Broad.	72nd Broad.	Times Sq.	Penn St.	86th Lex.	59th Lex.	Grand Centr.
dist			2	1	1			1	0	1		2	1
prev			59th Broad.	Times Sq.	Times Sq.			Times Sq.	source	Times Sq.		Grand Centr.	Times Sq.



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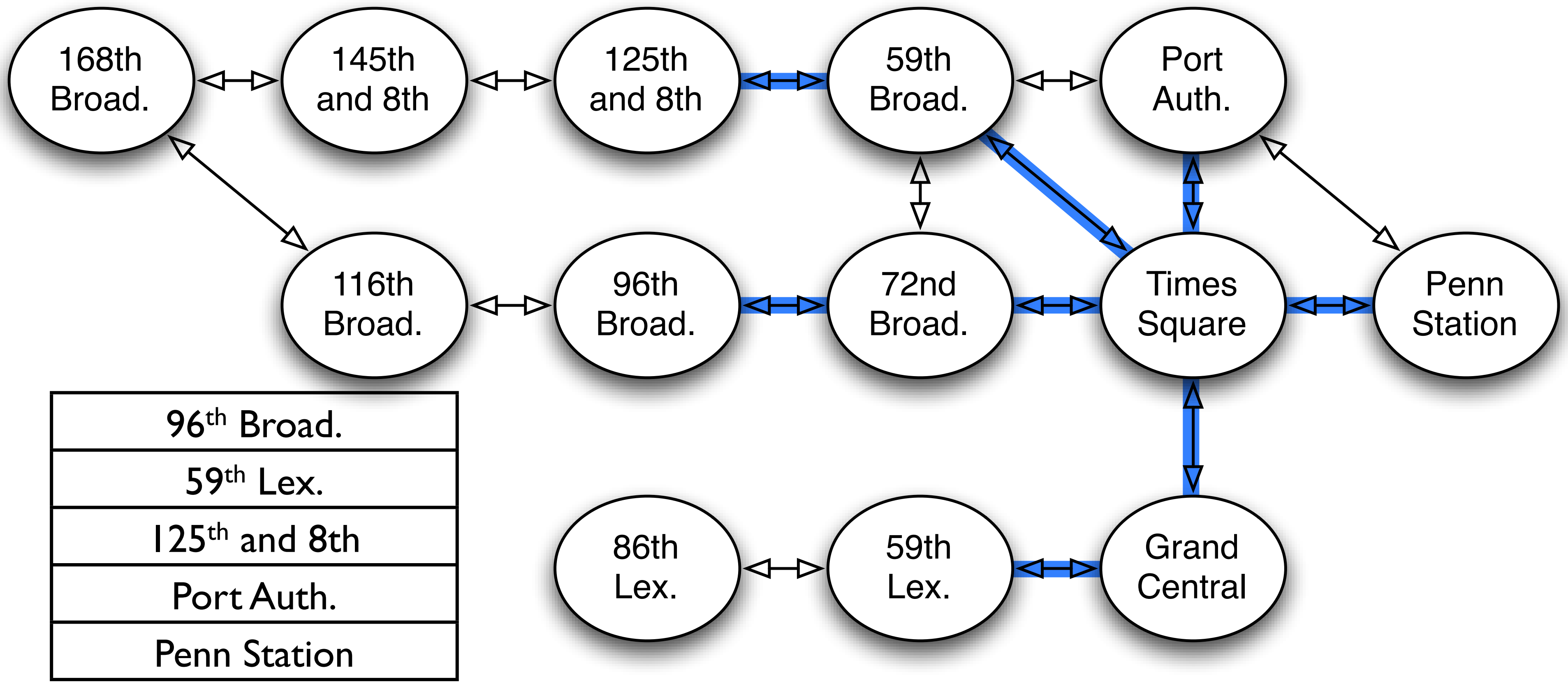
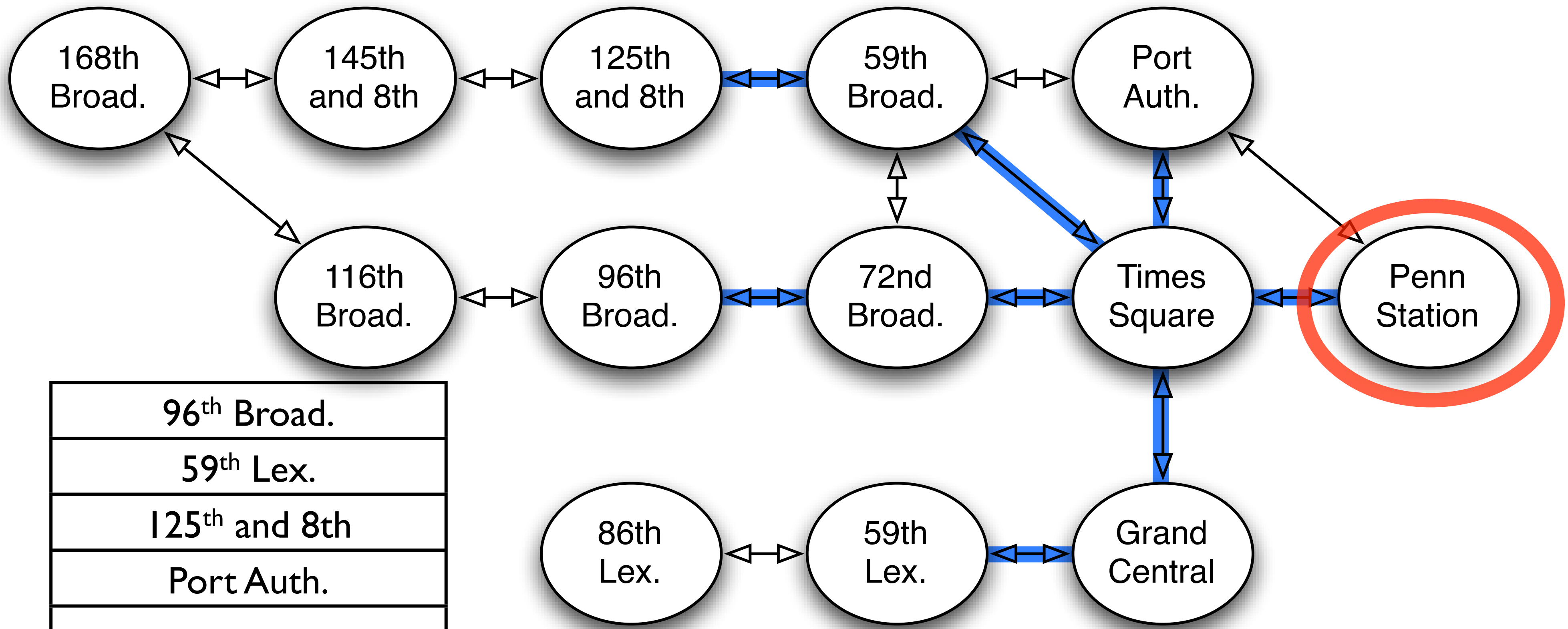
dist			2	1	1			1	0	1		2	1
prev			59th Broad.	Times Sq.	Times Sq.			Times Sq.	source	Times Sq.		Grand Centr.	Times Sq.

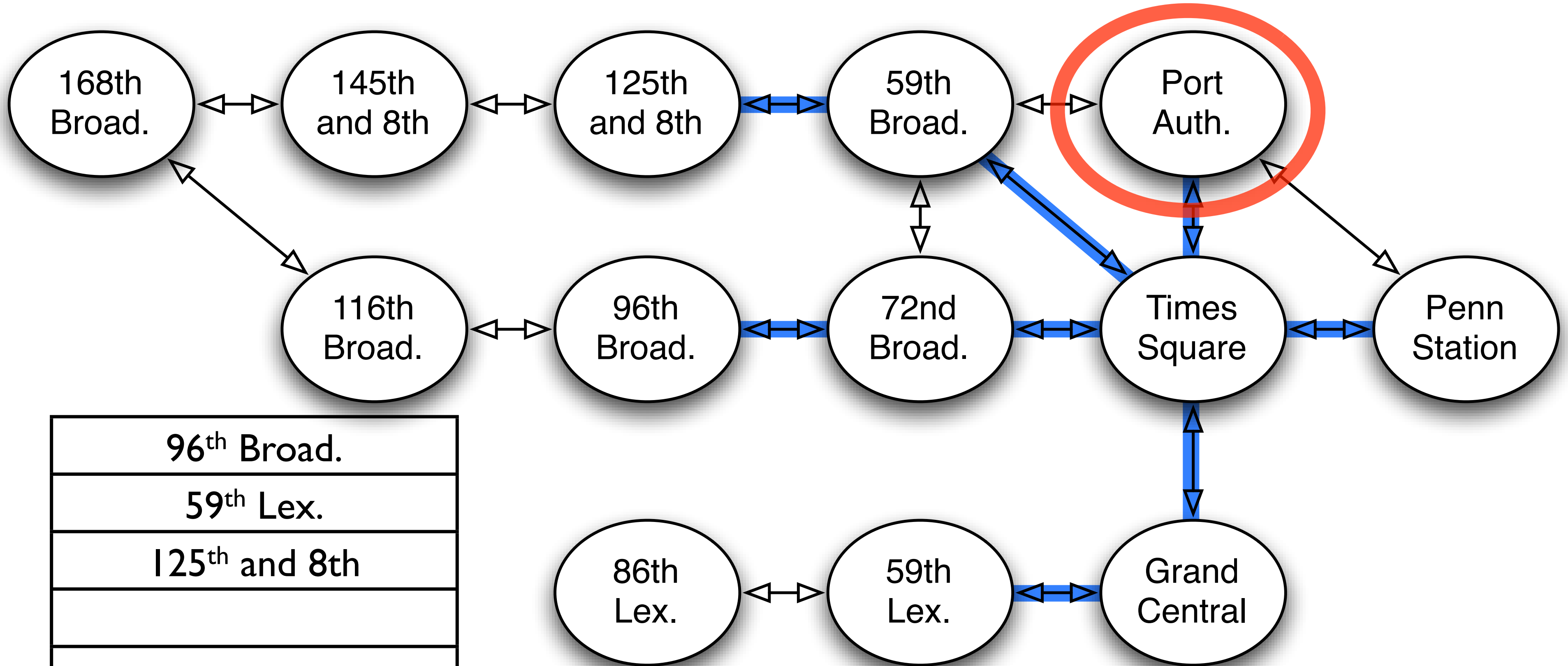


	168th Broad.	145th Broad.	125th 8th	59th Broad.	Port Auth.	116th Broad.	96th Broad.	72nd Broad.	Times Sq.	Penn St.	86th Lex.	59th Lex.	Grand Centr.
dist			2	1	1		2	1	0	1		2	1
prev			59th Broad.	Times Sq.	Times Sq.		72nd Broad.	Times Sq.	source	Times Sq.		Grand Centr.	Times Sq.



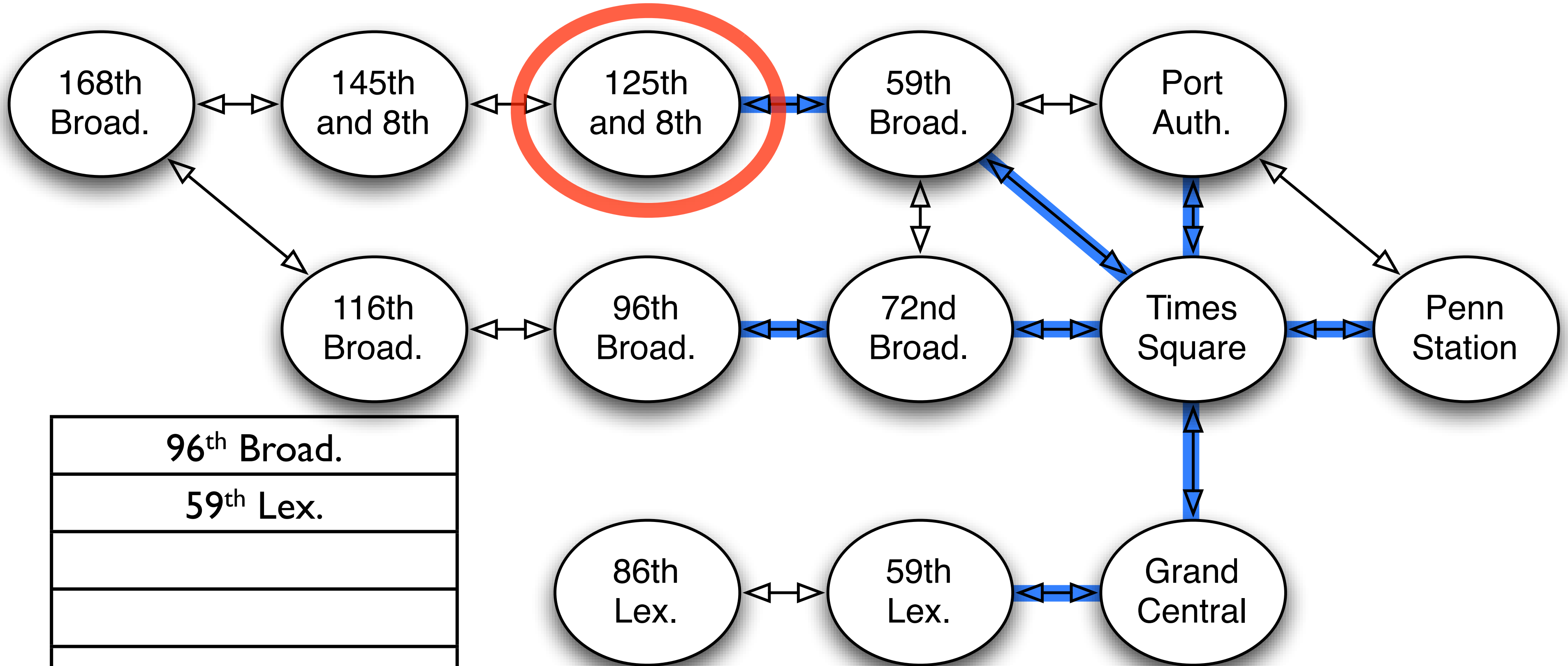
96th Broad.
59th Lex.
125th and 8th
Port Auth.

	168th Broad.	145th Broad.	125th 8th	59th Broad.	Port Auth.	116th Broad.	96th Broad.	72nd Broad.	Times Sq.	Penn St.	86th Lex.	59th Lex.	Grand Centr.
dist			2	1	1		2	1	0	1		2	1
prev			59th Broad.	Times Sq.	Times Sq.		72nd Broad.	Times Sq.	source	Times Sq.		Grand Centr.	Times Sq.



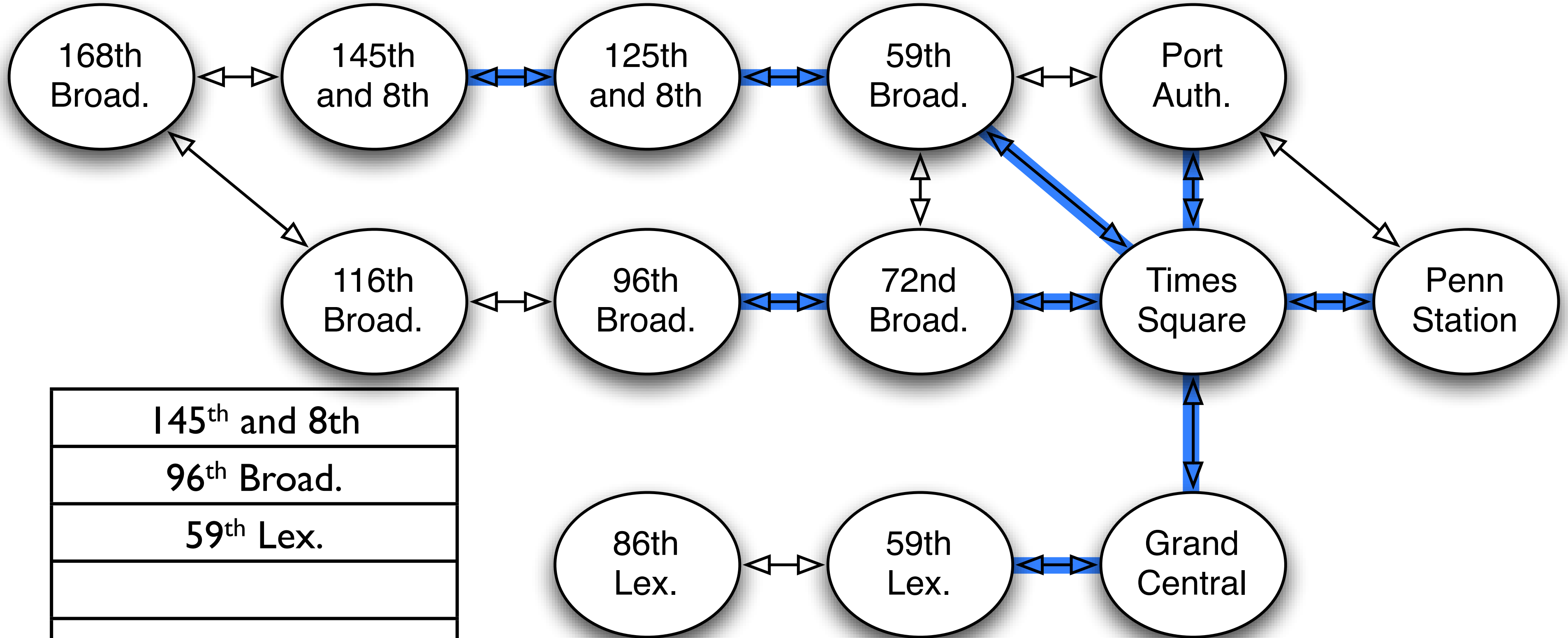
96th Broad.
59th Lex.
125th and 8th

	168th Broad.	145th Broad.	125th 8th	59th Broad.	Port Auth.	116th Broad.	96th Broad.	72nd Broad.	Times Sq.	Penn St.	86th Lex.	59th Lex.	Grand Centr.
dist			2	1	1		2	1	0	1		2	1
prev			59th Broad.	Times Sq.	Times Sq.		72nd Broad.	Times Sq.	source	Times Sq.		Grand Centr.	Times Sq.



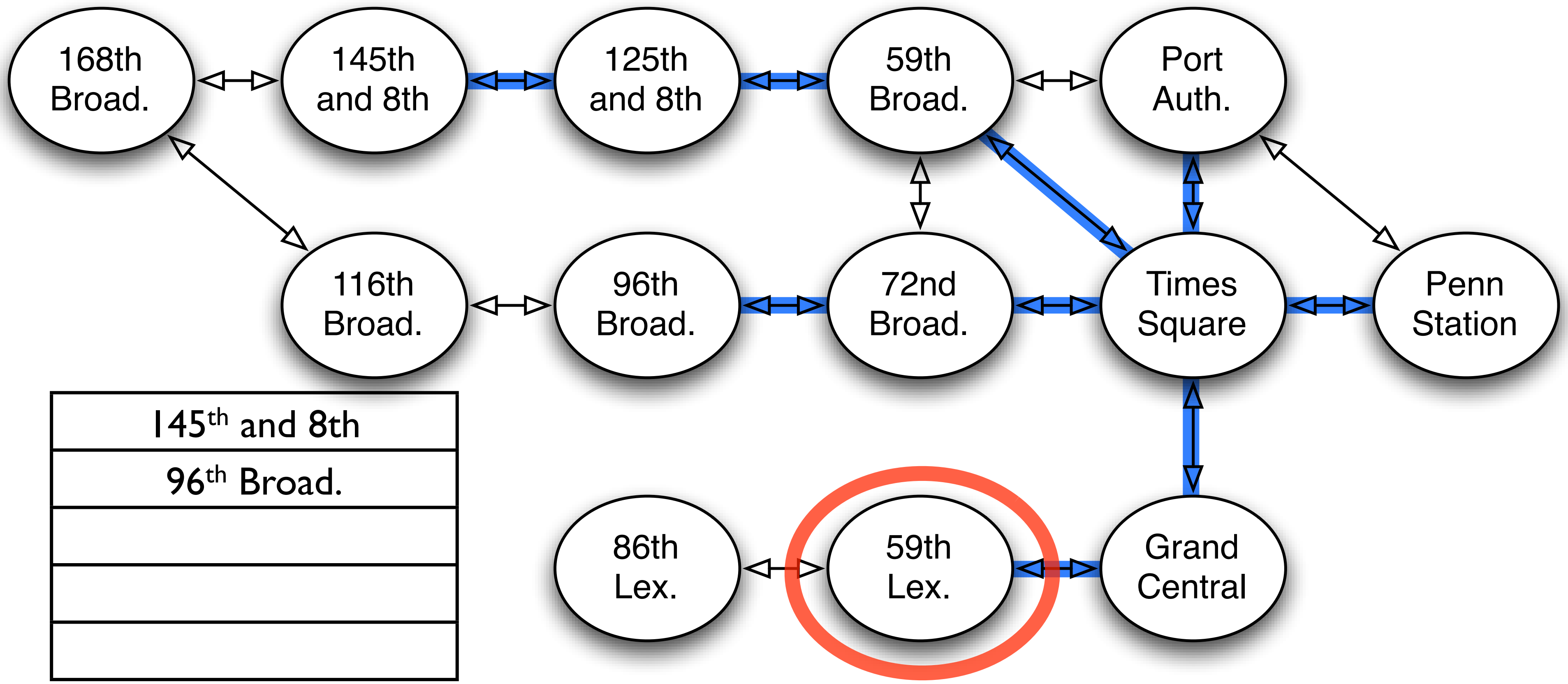
96th Broad.
59th Lex.

	168th Broad.	145th Broad.	125th 8th	59th Broad.	Port Auth.	116th Broad.	96th Broad.	72nd Broad.	Times Sq.	Penn St.	86th Lex.	59th Lex.	Grand Centr.
dist			2	1	1		2	1	0	1		2	1
prev			59th Broad.	Times Sq.	Times Sq.		72nd Broad.	Times Sq.	source	Times Sq.		Grand Centr.	Times Sq.



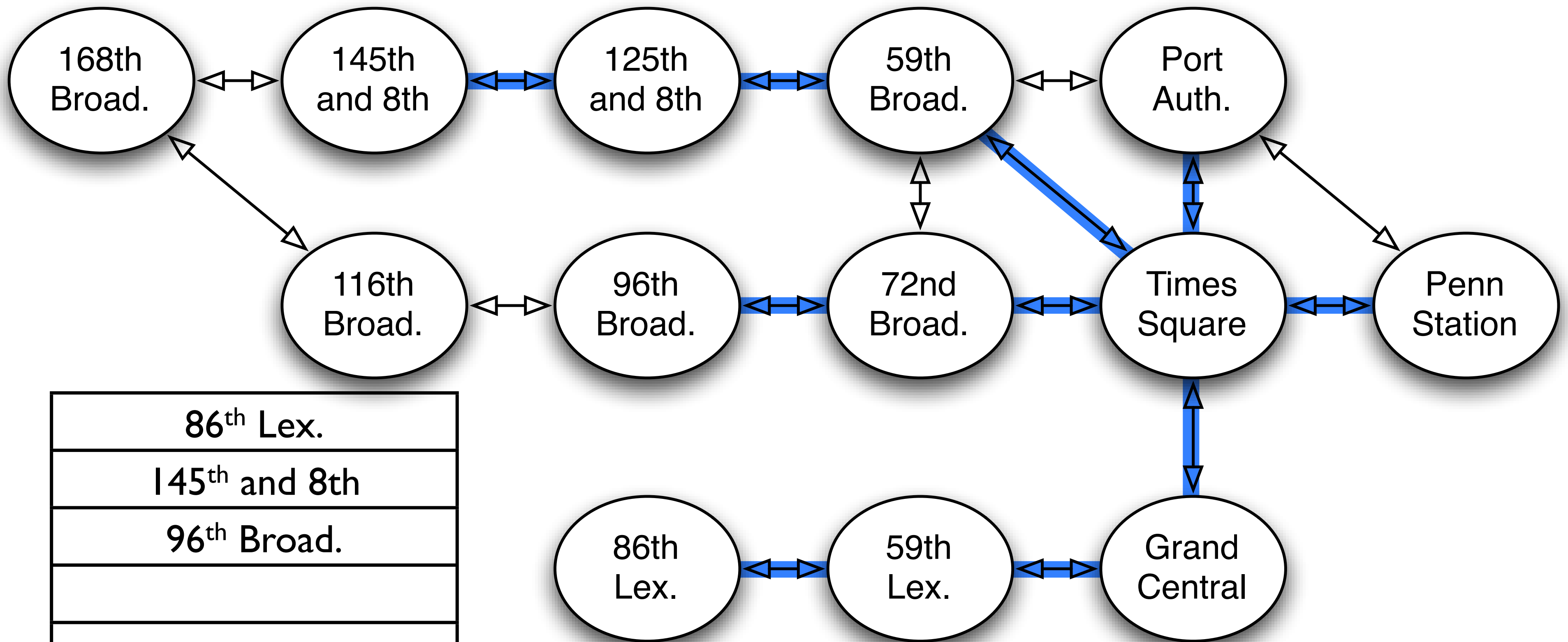
145th and 8th
96th Broad.
59th Lex.

	168th Broad.	145th Broad.	125th 8th	59th Broad.	Port Auth.	116th Broad.	96th Broad.	72nd Broad.	Times Sq.	Penn St.	86th Lex.	59th Lex.	Grand Centr.
dist		3	2	1	1		2	1	0	1		2	1
prev		125th 8th	59th Broad.	Times Sq.	Times Sq.		72nd Broad.	Times Sq.	source	Times Sq.		Grand Centr.	Times Sq.



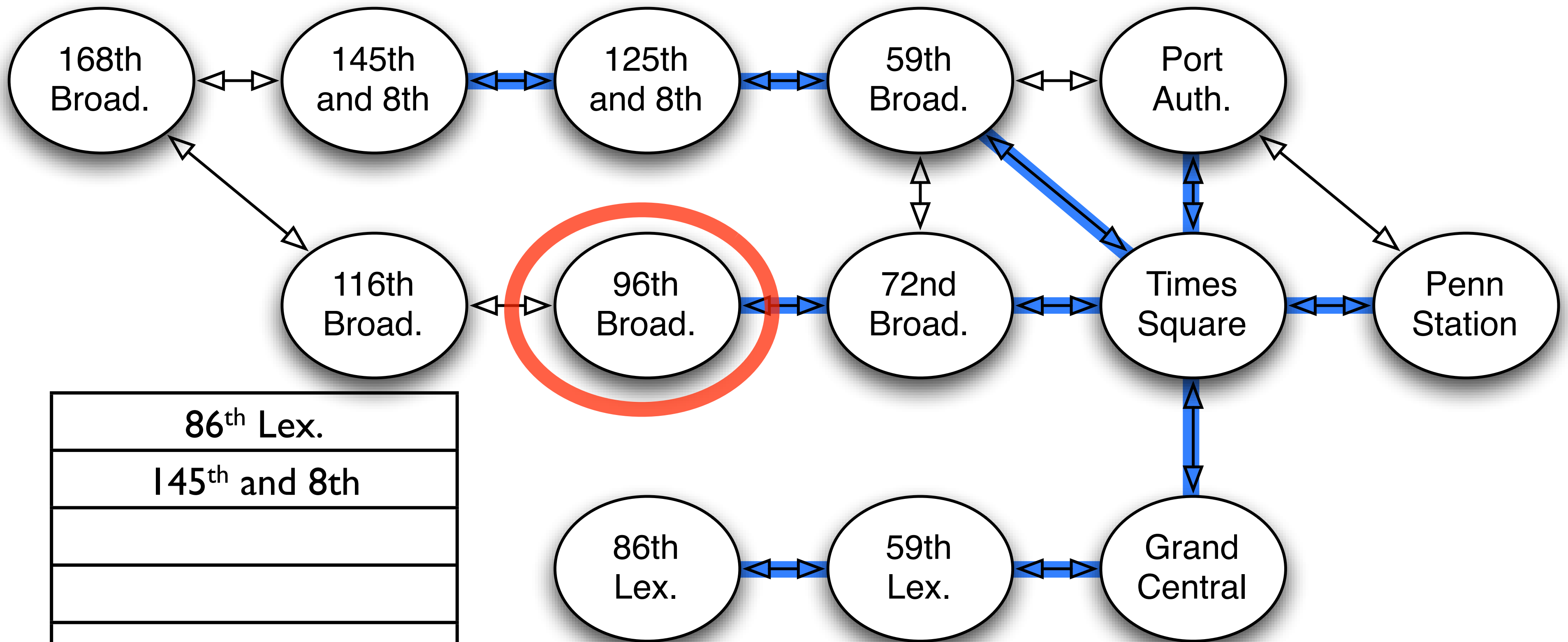
145th and 8th
96th Broad.

	168th Broad.	145th Broad.	125th 8th	59th Broad.	Port Auth.	116th Broad.	96th Broad.	72nd Broad.	Times Sq.	Penn St.	86th Lex.	59th Lex.	Grand Centr.
dist		3	2	1	1		2	1	0	1		2	1
prev		125th 8th	59th Broad.	Times Sq.	Times Sq.		72nd Broad.	Times Sq.	source	Times Sq.		Grand Centr.	Times Sq.



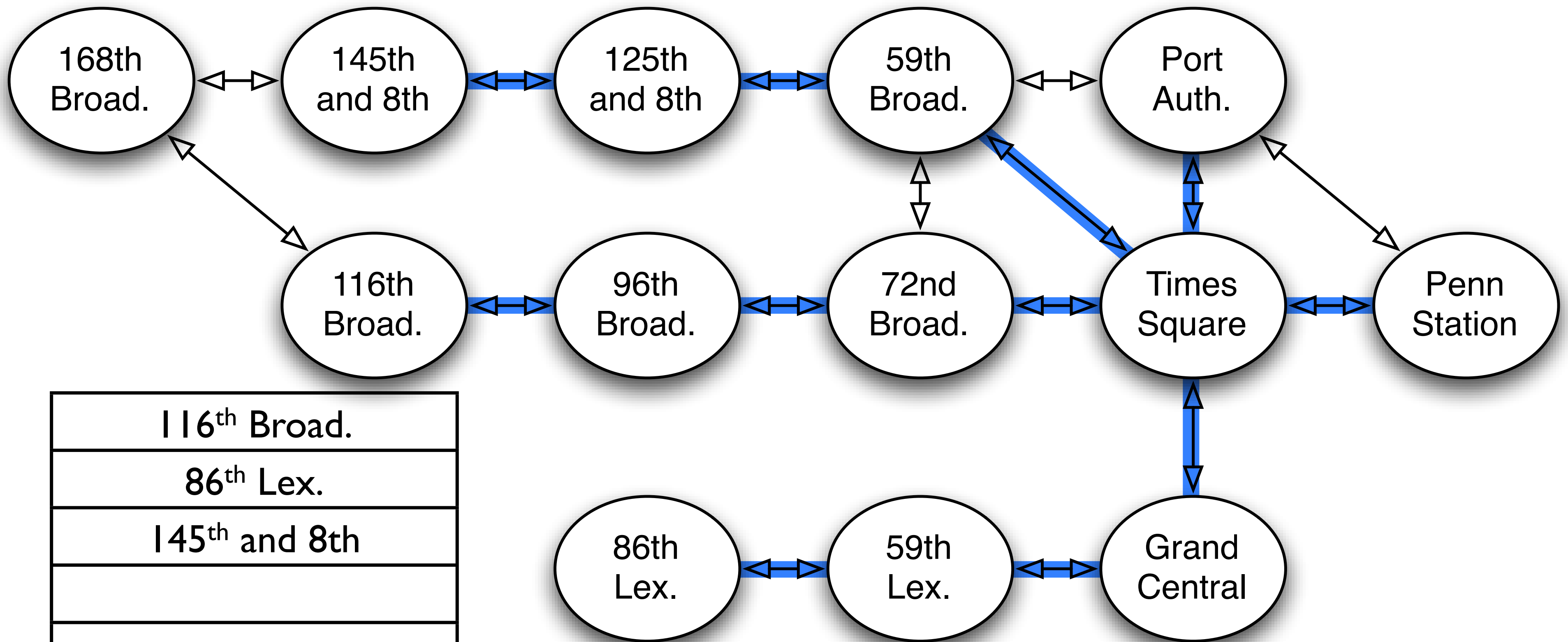
86th Lex.
145th and 8th
96th Broad.

	168th Broad.	145th Broad.	125th 8th	59th Broad.	Port Auth.	116th Broad.	96th Broad.	72nd Broad.	Times Sq.	Penn St.	86th Lex.	59th Lex.	Grand Centr.
dist		3	2	1	1		2	1	0	1	3	2	1
prev		125th 8th	59th Broad.	Times Sq.	Times Sq.		72nd Broad.	Times Sq.	source	Times Sq.	59th Lex.	Grand Centr.	Times Sq.



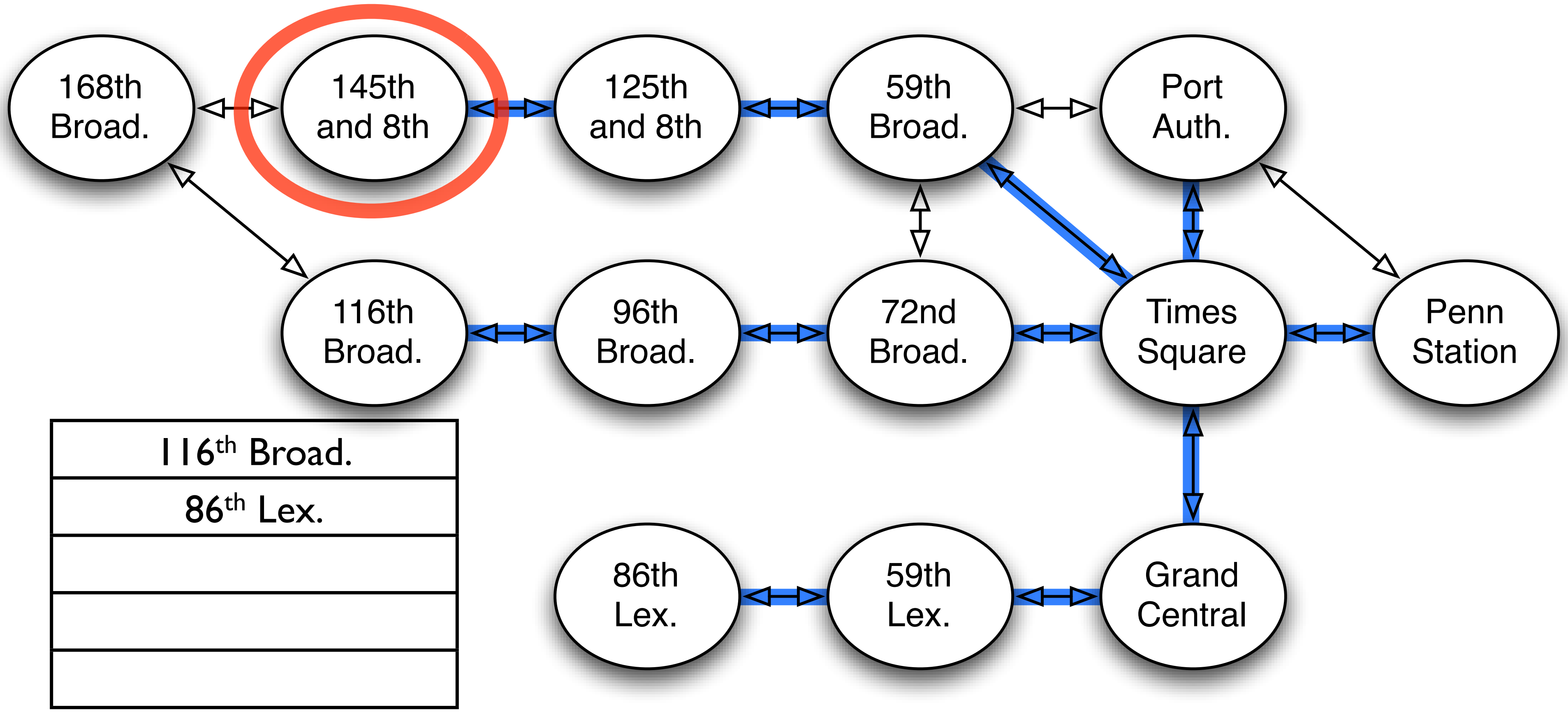
86th Lex.
145th and 8th

	168th Broad.	145th Broad.	125th 8th	59th Broad.	Port Auth.	116th Broad.	96th Broad.	72nd Broad.	Times Sq.	Penn St.	86th Lex.	59th Lex.	Grand Centr.
dist		3	2	1	1		2	1	0	1	3	2	1
prev		125th 8th	59th Broad.	Times Sq.	Times Sq.		72nd Broad.	Times Sq.	source	Times Sq.	59th Lex.	Grand Centr.	Times Sq.

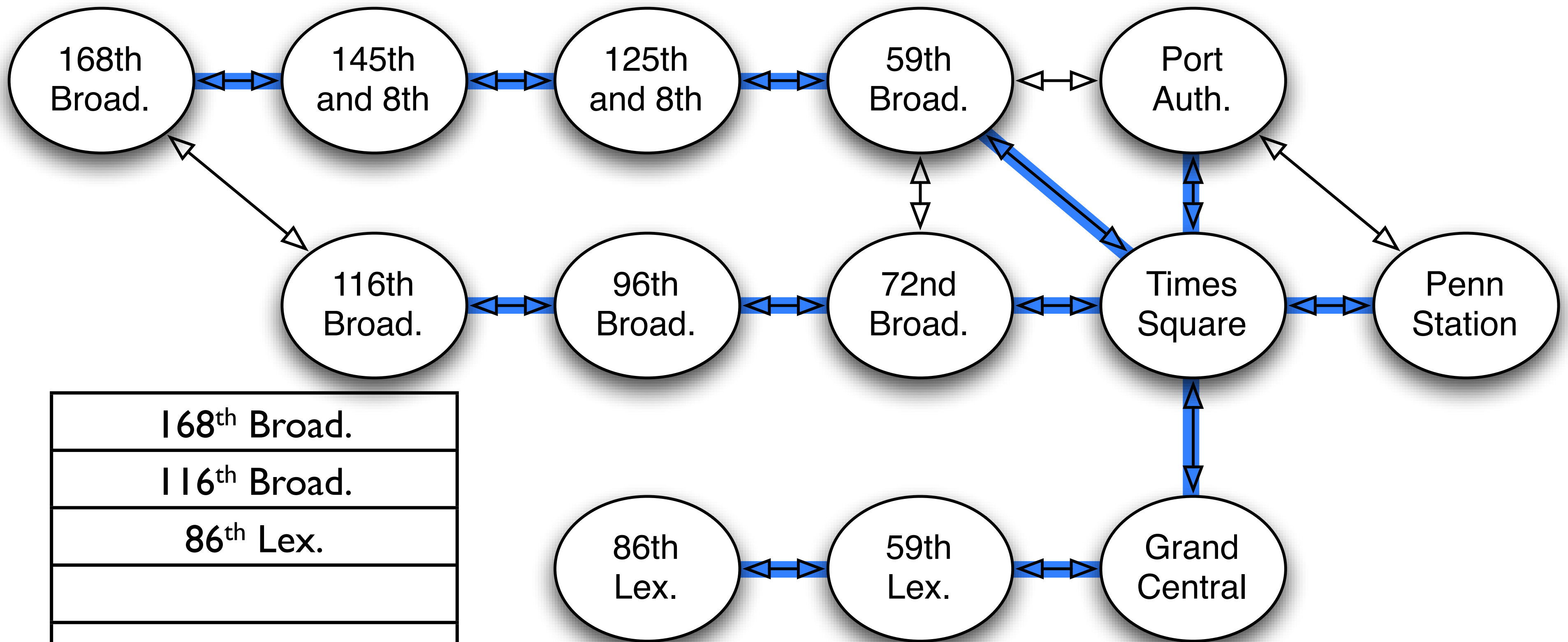


116th Broad.
86th Lex.
145th and 8th

	168th Broad.	145th Broad.	125th 8th	59th Broad.	Port Auth.	116th Broad.	96th Broad.	72nd Broad.	Times Sq.	Penn St.	86th Lex.	59th Lex.	Grand Centr.
dist		3	2	1	1	3	2	1	0	1	3	2	1
prev		125th 8th	59th Broad.	Times Sq.	Times Sq.	96th Broad.	72nd Broad.	Times Sq.	source	Times Sq.	59th Lex.	Grand Centr.	Times Sq.



	168th Broad.	145th Broad.	125th 8th	59th Broad.	Port Auth.	116th Broad.	96th Broad.	72nd Broad.	Times Sq.	Penn St.	86th Lex.	59th Lex.	Grand Centr.
dist		3	2	1	1	3	2	1	0	1	3	2	1
prev		125th 8th	59th Broad.	Times Sq.	Times Sq.	96th Broad.	72nd Broad.	Times Sq.	source	Times Sq.	59th Lex.	Grand Centr.	Times Sq.



168th Broad.
116th Broad.
86th Lex.

	168th Broad.	145th Broad.	125th 8th	59th Broad.	Port Auth.	116th Broad.	96th Broad.	72nd Broad.	Times Sq.	Penn St.	86th Lex.	59th Lex.	Grand Centr.
dist	4	3	2	1	1	3	2	1	0	1	3	2	1
prev	145th 8th	125th 8th	59th Broad.	Times Sq.	Times Sq.	96th Broad.	72nd Broad.	Times Sq.	source	Times Sq.	59th Lex.	Grand Centr.	Times Sq.

Weighted Shortest Path

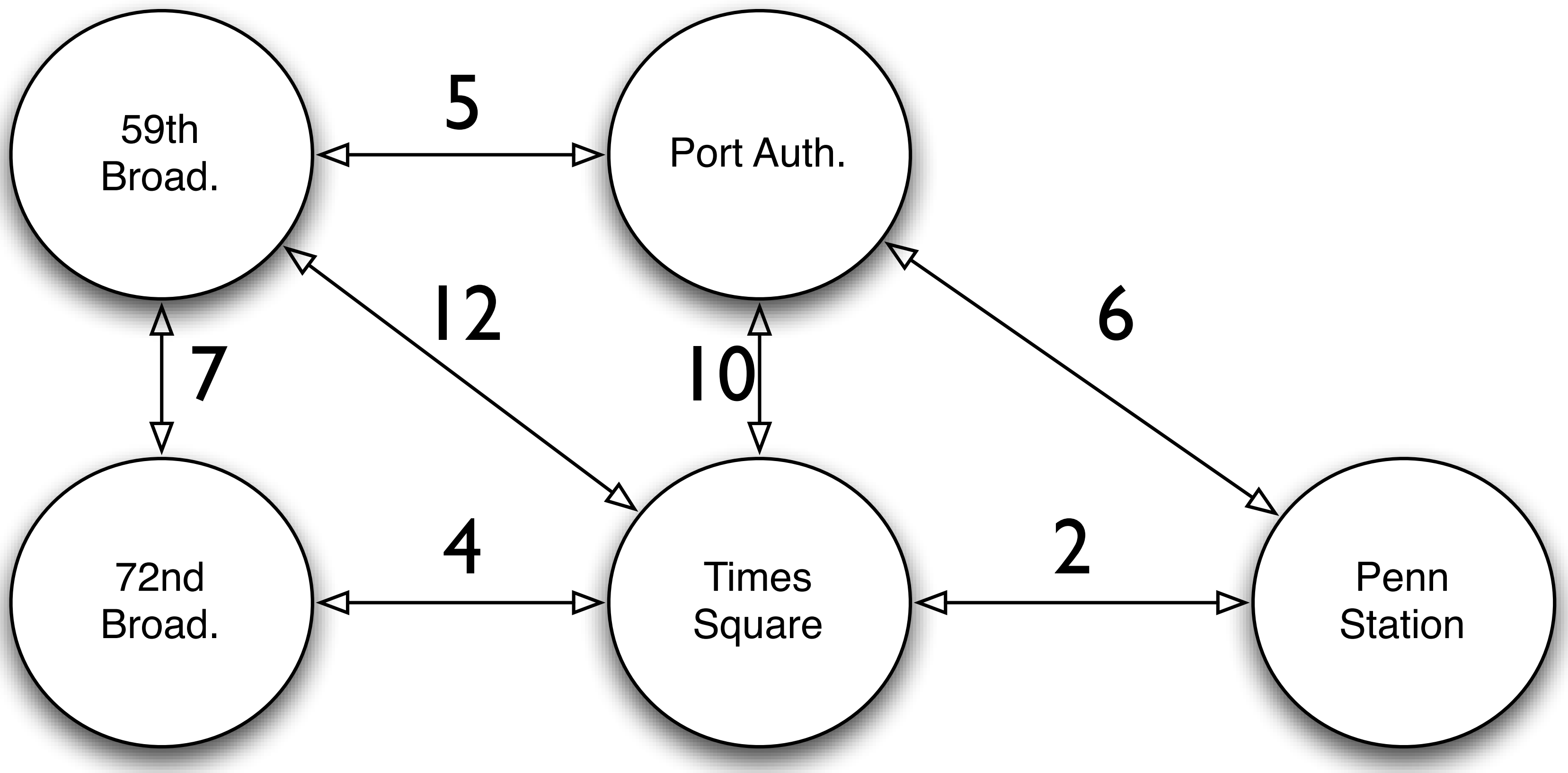
- The problem becomes more difficult when edges have different weights
- Weights represent different costs on using that edge
- Standard algorithm is **Dijkstra's Algorithm**

Dijkstra’s Algorithm

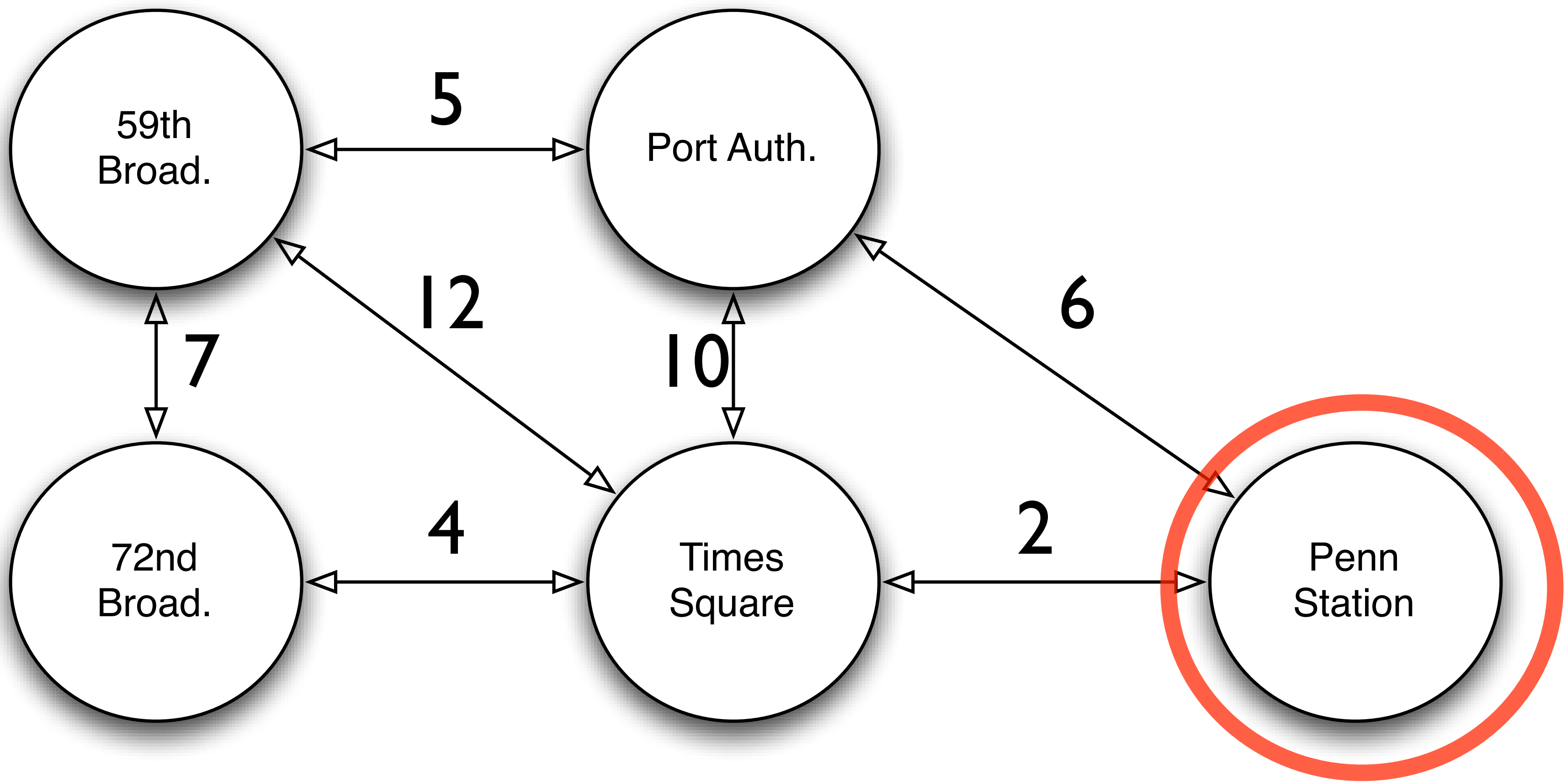
- Keep distance overestimates **D(v)** for each node **v** (all non-source nodes are initially infinite)
- 1. Choose node **v** with smallest *unknown* distance
- 2. Declare that **v**’s shortest distance is *known*
- 3. Update distance estimates for neighbors

Updating Distances

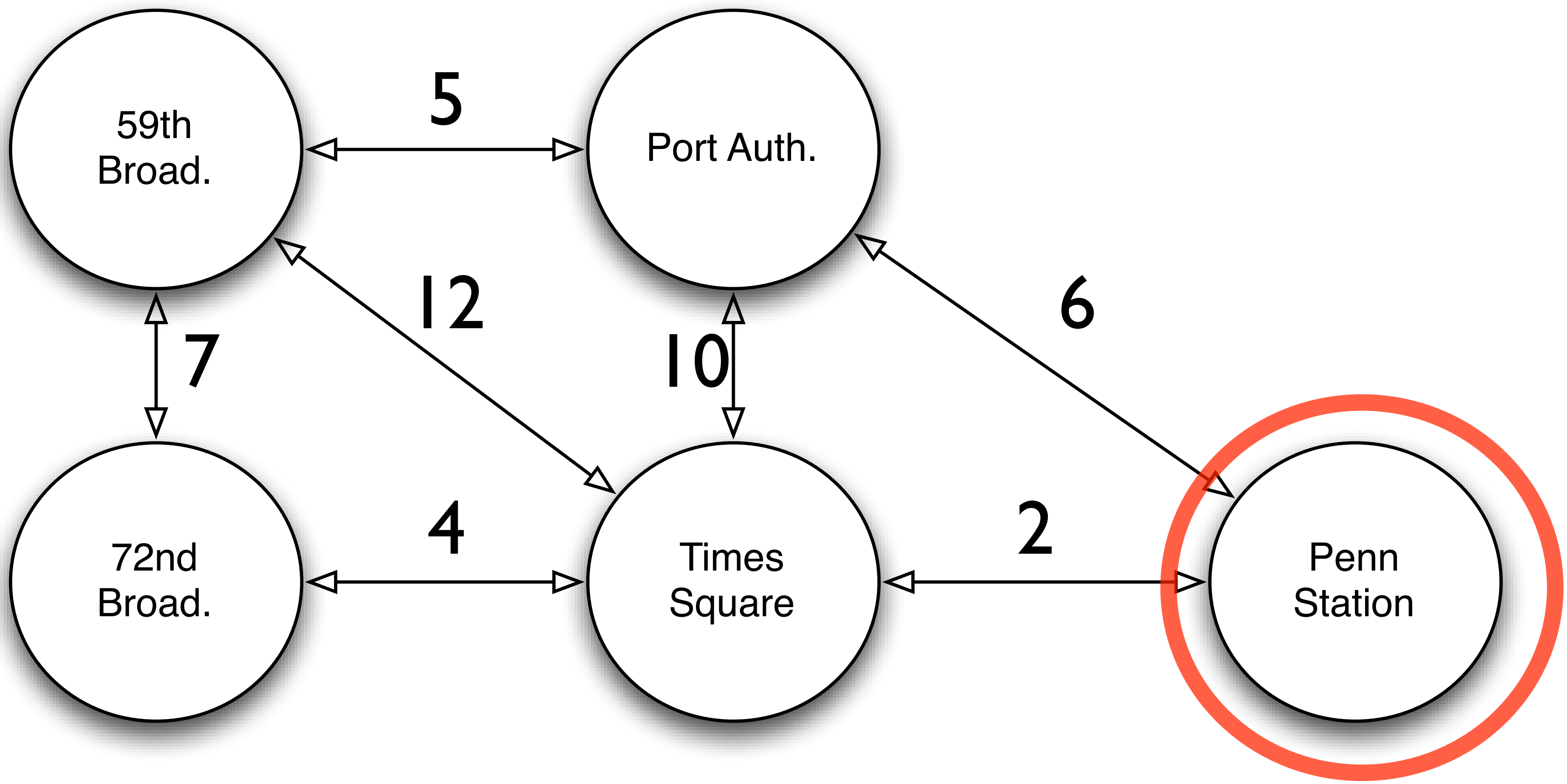
- For each of **v**'s neighbors, **w**,
- if min(**D(v)+ weight(v,w)**, **D(w)**)
    - i.e., update **D(w)** if the path going through **v** is cheaper than the best path so far to **w**



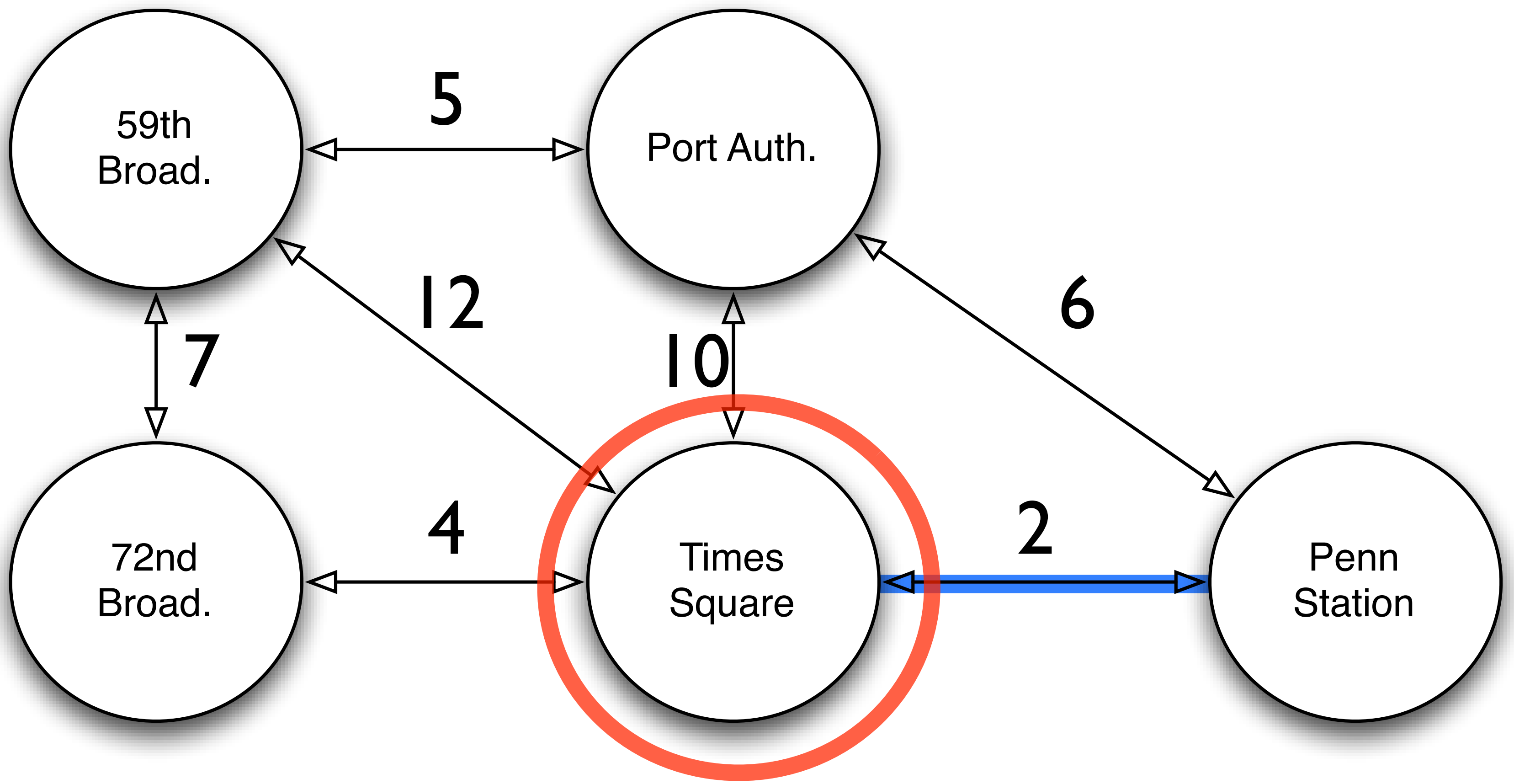
59th Broad.	Port Auth.	72nd Broad	Times Sq.	Penn St.
inf	inf	inf	inf	0
?	?	?	?	home



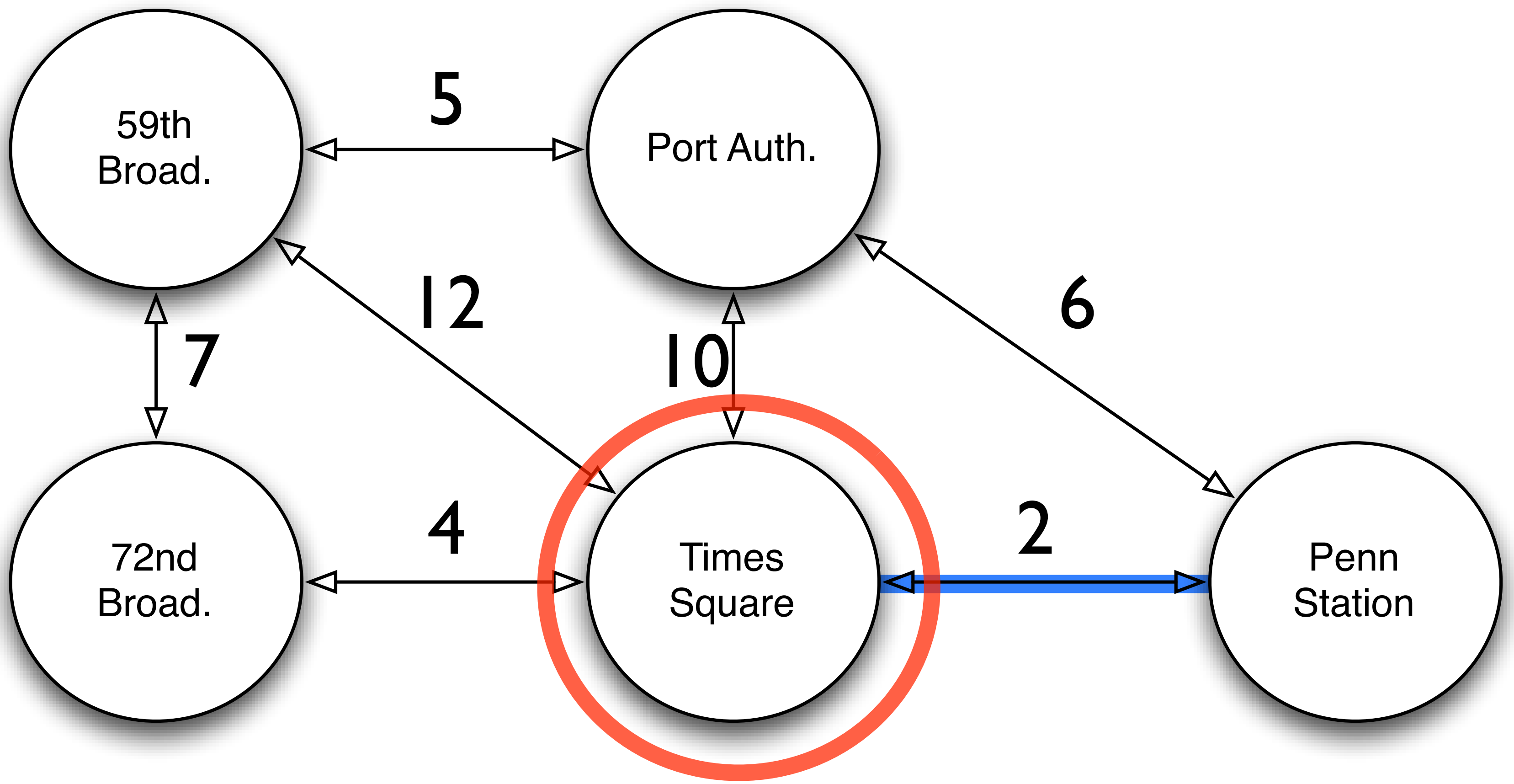
59th Broad.	Port Auth.	72nd Broad	Times Sq.	Penn St.
inf	inf	inf	inf	**0**
?	?	?	?	**home**



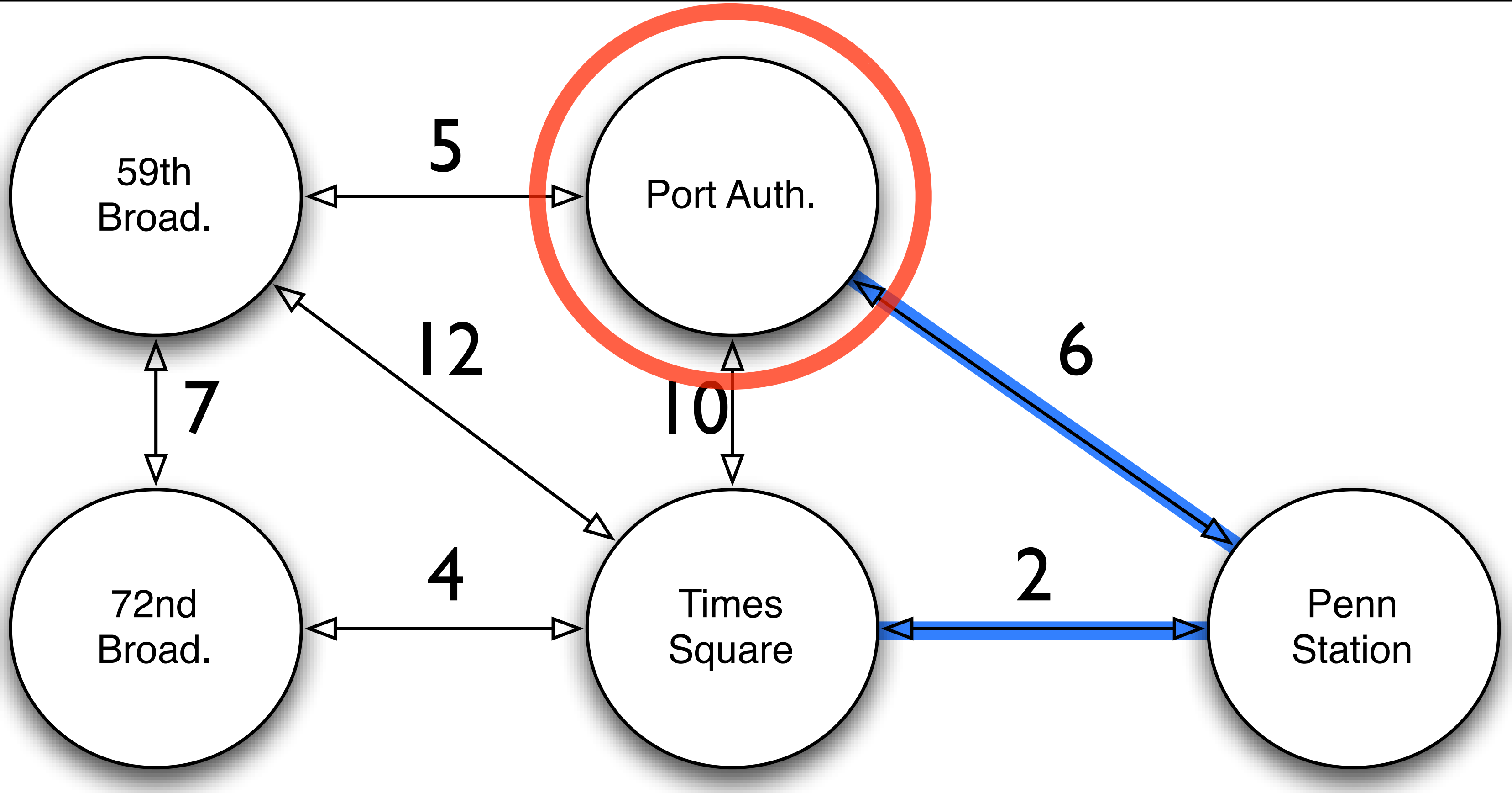
59th Broad.	Port Auth.	72nd Broad	Times Sq.	Penn St.
inf	6	inf	2	**0**
?	Penn St.?	?	Penn St.?	**home**



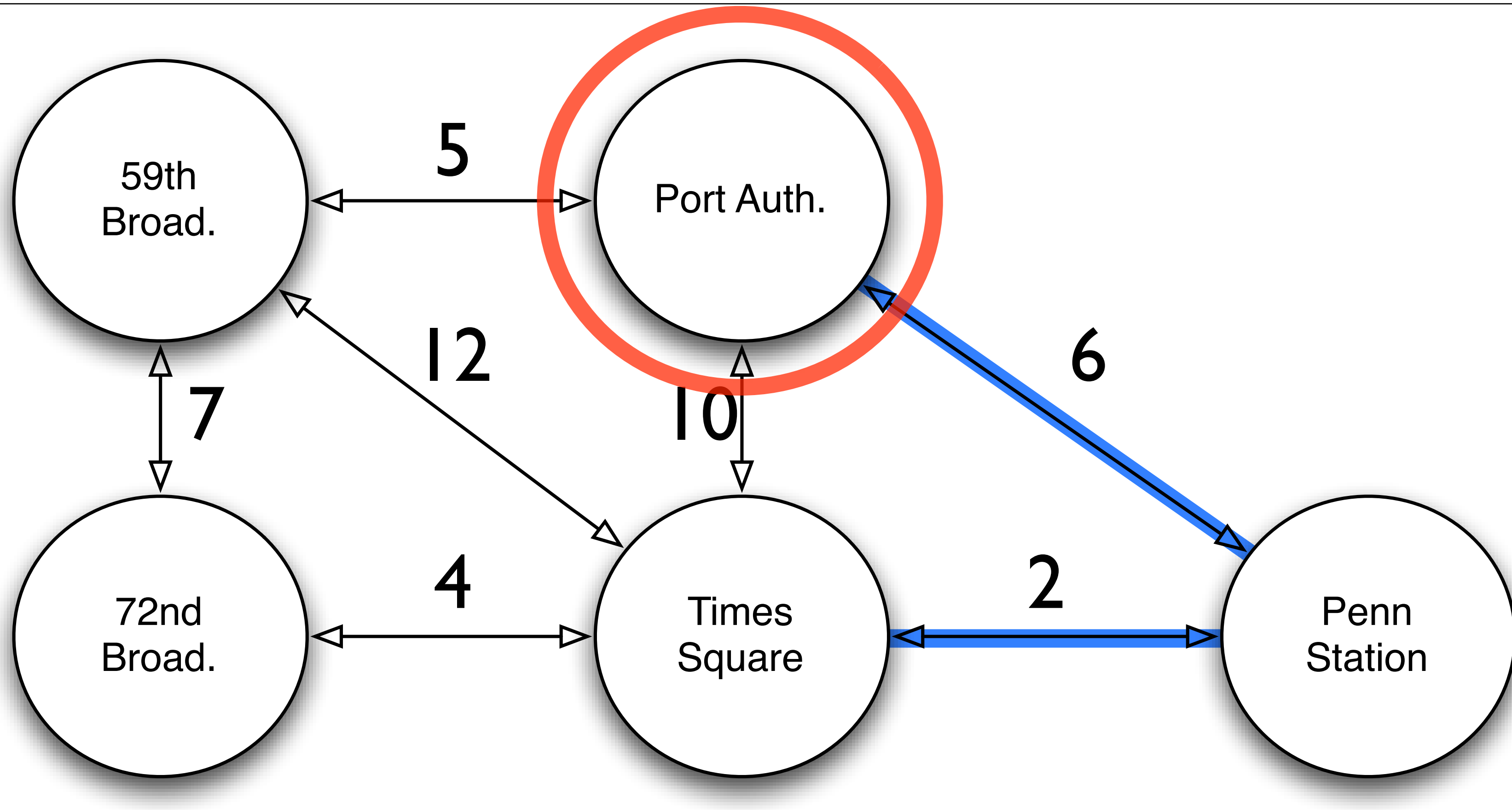
59th Broad.	Port Auth.	72nd Broad	Times Sq.	Penn St.
inf	6	inf	**2**	**0**
?	Penn St.?	?	**Penn St.**	**home**



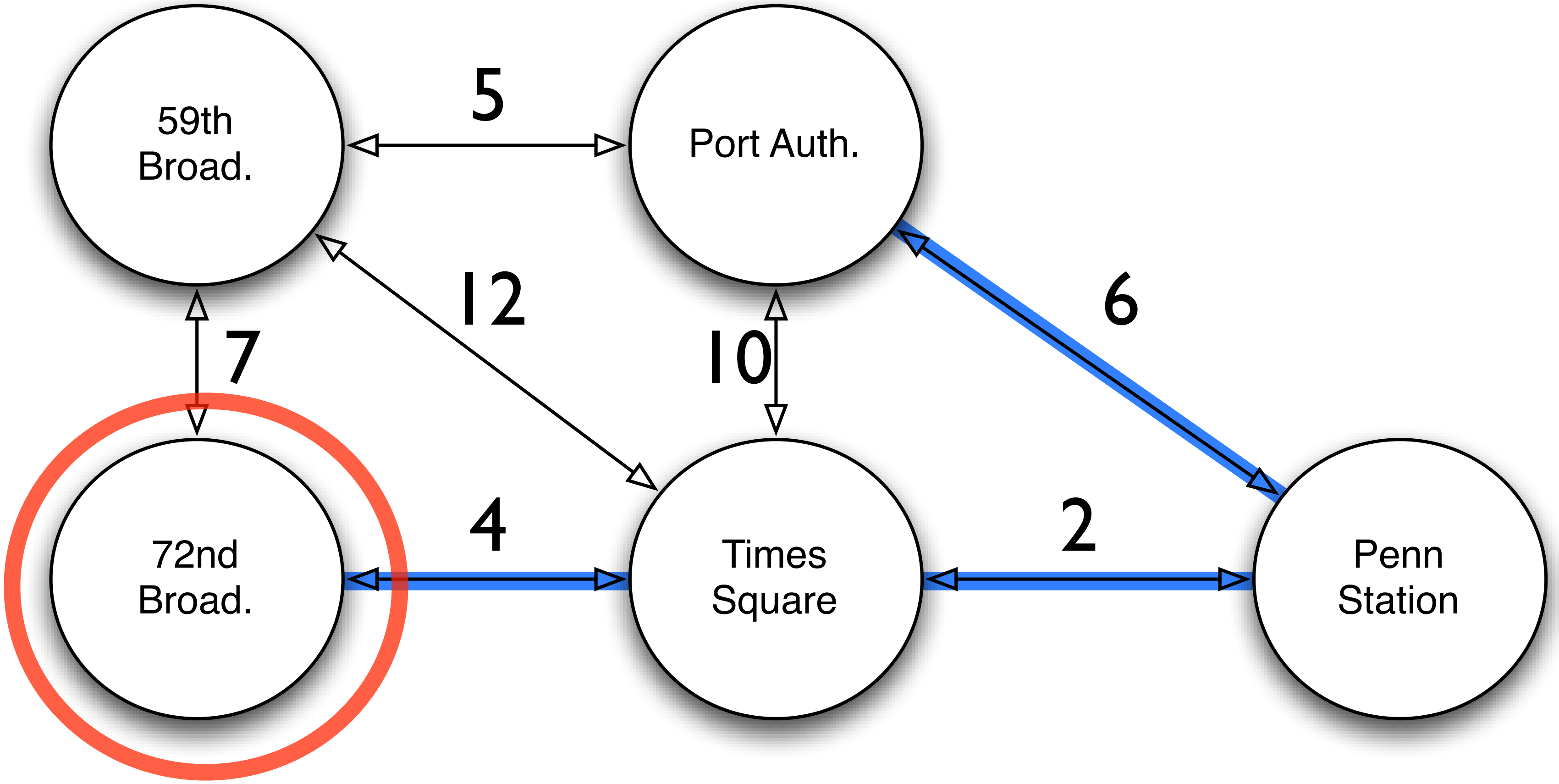
59th Broad.	Port Auth.	72nd Broad	Times Sq.	Penn St.
2+12=14	6	2+4=6	**2**	**0**
Times Sq?	Penn St.?	Times Sq?	**Penn St.**	**home**



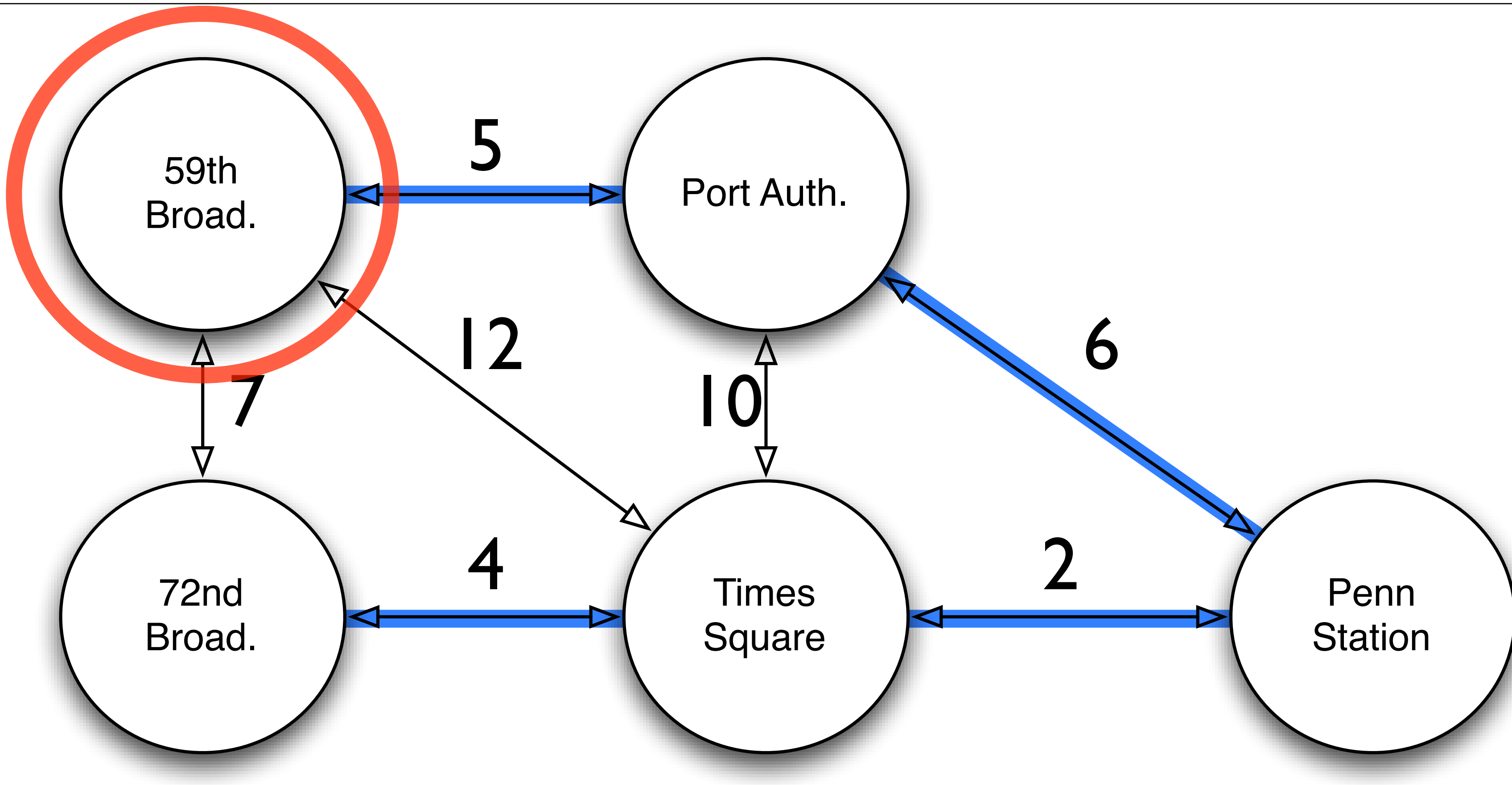
59th Broad.	Port Auth.	72nd Broad	Times Sq.	Penn St.
14	**6**	6	**2**	**0**
Times Sq?	**Penn St.**	Times Sq?	**Penn St.**	**home**



59th Broad.	Port Auth.	72nd Broad	Times Sq.	Penn St.
5+6=11	**6**	6	**2**	**0**
Port Auth?	**Penn St.**	Times Sq?	**Penn St.**	**home**



59th Broad.	Port Auth.	72nd Broad	Times Sq.	Penn St.
11	**6**	**6**	**2**	**0**
Port Auth?	**Penn St.**	**Times Sq**	**Penn St.**	**home**



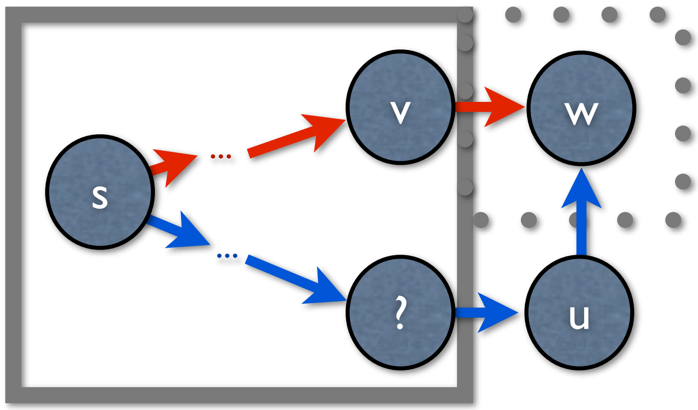
59th Broad.	Port Auth.	72nd Broad	Times Sq.	Penn St.
11	**6**	**6**	**2**	**0**
Port Auth	**Penn St.**	**Times Sq**	**Penn St.**	**home**

Dijkstra's Algorithm Analysis

- First, convince ourselves that the algorithm works.
- At each stage, we have a set of nodes whose shortest paths we know
- In the base case, the set is the source node.
- Inductive step: if we have a correct set, is greedily adding the shortest neighbor correct?

Proof by Contradiction (Sketch)

- Contradiction: Dijkstra's finds a shortest path to node **w** through **v**, but there exists an even shorter path
- This shorter path must pass from inside our known set to outside.
- Call the 1st node in cheaper path outside our set **u**



- The path to **u** must be shorter than the path to **w**
  - But then we would have chosen **u** instead

Computational Cost

- If the graph is dense, we scan the vertices to find the minimum edge **O(V)**
- This happens **|V|** times
- We also update the distances once per edge, **O(|E|)**
- Thus, total running time is $O(|E| + |V|^2)$

Computational Cost (sparse)

- Keep a priority queue of all unknown nodes
- Each stage requires a **deleteMin**, and then some **decreaseKey**s (the # of neighbors of node)
- We call **decreaseKey** once per edge, we call **deleteMin** once per vertex
- Both operations are $O(\log |V|)$
- Total cost: $O(|E| \log |V| + |V| \log |V|) = O(|E| \log |V|)$

All Pairs Shortest Path

- Dijkstra's Algorithm finds shortest paths from one node to all other nodes
- What about computing shortest paths for all pairs of nodes?
- We can run Dijkstra's |V| times. Total cost: $O(|V|^3)$
- Floyd-Warshall algorithm is often faster in practice (though same asymptotic time)

Recursive Motivation

- Consider the set of numbered nodes **1** through **k**
- The shortest path between any node **i** and **j** using only nodes in the set {**1**, ..., **k**} is the minimum of
  - shortest path from **i** to **j** using nodes {**1**, ..., **k-1**}
  - shortest path from **i** to **j** using node **k**
- dist(i,j,k) = min(dist(i,j,k-1),
  dist(i,k,k-1)+dist(k,j,k-1))

Dynamic Programming

- Instead of repeatedly computing recursive calls, store lookup table
- To compute dist(i,j,k) for any i,j, we only need to look up dist(-,-, k-1)
    - but never k-2, k-3, etc.
- We can incrementally compute the path matrix for k=0, then use it to compute for k=1, then k=2...

Floyd-Warshall Code

- Initialize `d = ` weight matrix
-
```
for (k=0; k<N; k++)
   for (i=0; i<N; i++)
      for (j=0; j<N; j++)
         if (d[i][j] > d[i][k]+d[k][j])
            d[i][j] = d[i][k] + d[k][j];
```
- Additionally, we can store the actual path by keeping a “midpoint” matrix

Midpoint Matrix

- We can store the N^2 paths efficiently with a midpoint matrix:

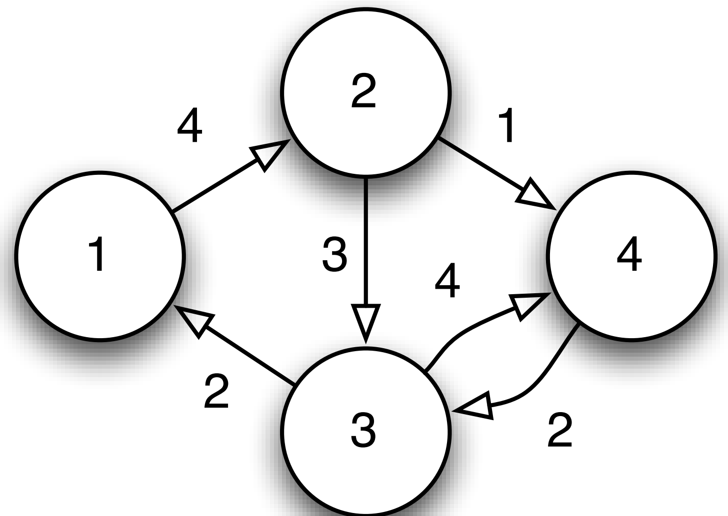
  path(i,j) = path(i, midpoint[i][j]) +
  path(midpoint[i][j], j)

- We only need a NxN matrix to store all the paths

All Pairs Shortest Path Example

k=0

	1	2	3	4
1	-	4	-	-
2	-	-	3	1
3	2	-	-	4
4	-	-	2	-



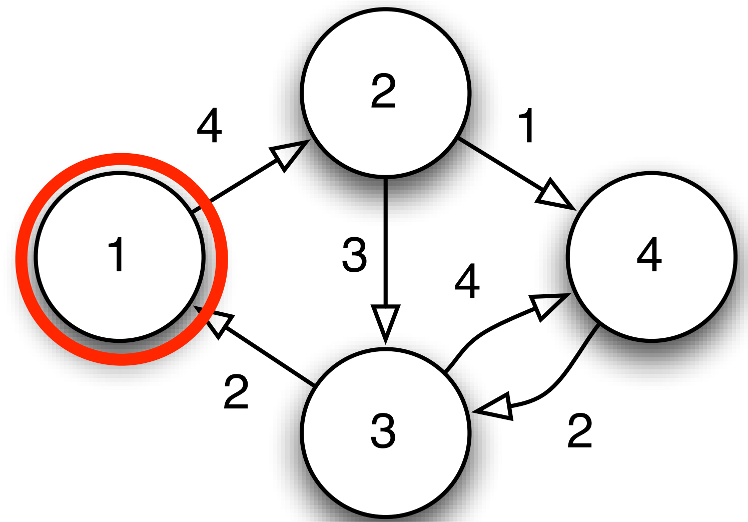
All Pairs Shortest Path Example

k=0

	1	2	3	4
1	-	4	-	-
2	-	-	3	1
3	2	-	-	4
4	-	-	2	-

k=1

	1	2	3	4
1	-	4	-	-
2	-	-	3	1
3	2	6	-	4
4	-	-	2	-



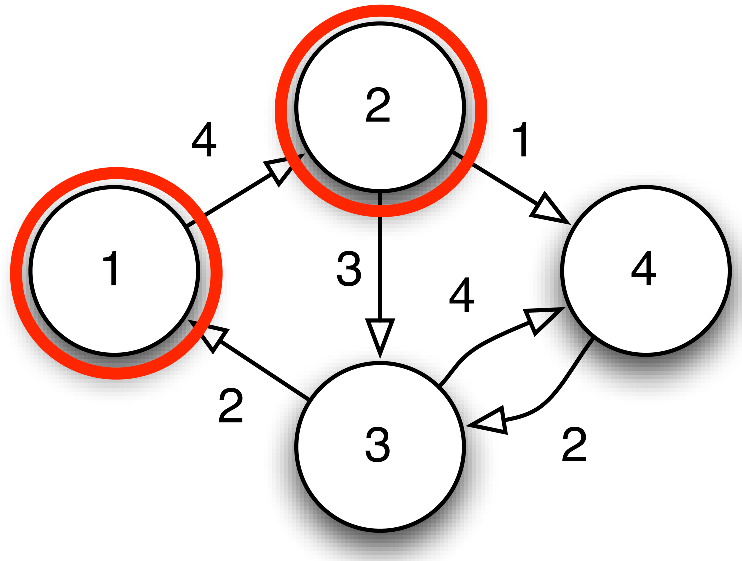
All Pairs Shortest Path Example

k=1

	1	2	3	4
1	-	4	-	-
2	-	-	3	1
3	2	6	-	4
4	-	-	2	-

k=2

	1	2	3	4
1	-	4	7	5
2	-	-	3	1
3	2	6	9	4
4	-	-	2	-



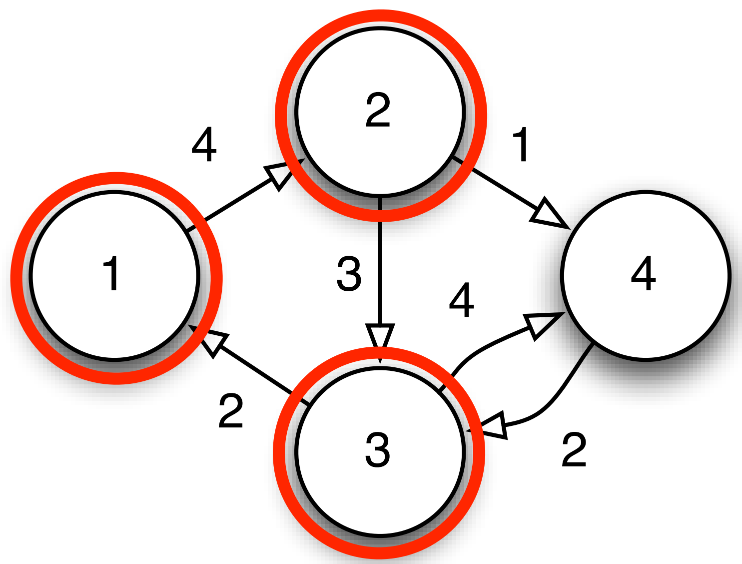
All Pairs Shortest Path Example

k=2

	1	2	3	4
1	-	4	7	5
2	-	-	3	1
3	2	6	9	4
4	-	-	2	-

k=3

	1	2	3	4
1	9	4	7	5
2	5	9	3	1
3	2	6	9	4
4	4	8	2	6



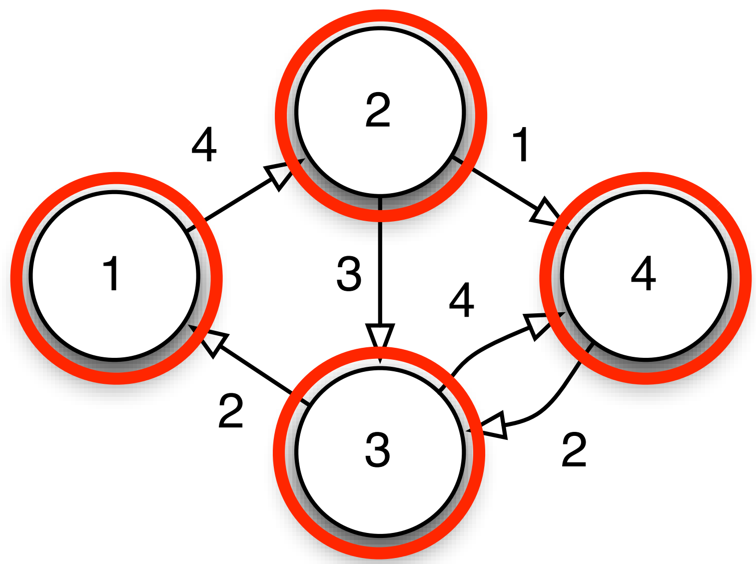
All Pairs Shortest Path Example

k=3

	1	2	3	4
1	9	4	7	5
2	5	9	3	1
3	2	6	9	4
4	4	8	2	6

k=4

	1	2	3	4
1	9	4	7	5
2	5	9	3	1
3	2	6	6	4
4	4	8	2	6



Transitive Closure

- For any nodes i, j, is there a path from i to j?
- Instead of computing shortest paths, just compute Boolean if a path exists
- path(i,j,k) = path(i,j,k-1) OR
  path(i,k,k-1) AND path(k,j,k-1)
- Transitive closure can tell you whether a graph is **connected**

Reading

- Weiss Section 9.1-9.3,
- Weiss Section 10.3.4 (All-Pairs Shortest Path)