Moreover,
\[ |\mathbf{a} \times \mathbf{b}|^2 (\mathbf{u} + \mathbf{v}) = [(\mathbf{b} \cdot \mathbf{a}) - (\mathbf{a} \cdot \mathbf{b})] \mathbf{u} - [(\mathbf{a} \cdot \mathbf{b}) \mathbf{a} - (\mathbf{a} \cdot \mathbf{a}) \mathbf{b}] \mathbf{v}, \]
and so
\[ |\mathbf{a} \times \mathbf{b}|^2 (\mathbf{u} + \mathbf{v}) \cdot \mathbf{c} = |\beta \mathbf{a} - \alpha \mathbf{b}|^2. \]
Now
\[ \mathbf{x} \cdot \mathbf{x} = |\mathbf{u} + \mathbf{v}|^2 + w^2 |\mathbf{a} \times \mathbf{b}|^2, \]
or
\[ |\mathbf{a} \times \mathbf{b}|^2 w^2 = |\mathbf{a} \times \mathbf{b}|^2 \gamma - |\beta \mathbf{a} - \alpha \mathbf{b}|^2. \]
We thus obtain the two solutions
\[ w = \pm \sqrt{|\mathbf{a} \times \mathbf{b}|^2 \gamma - |\beta \mathbf{a} - \alpha \mathbf{b}|^2}. \]

\[ \text{A.3 Vector and Matrix Differentiation} \]

Often a set of equations can be written more compactly in vector notation. The advantage of this may evaporate when it becomes necessary to look at the derivatives of a scalar or vector with respect to the components of a vector. It is, however, possible to use a consistent, compact notation in this case also.

\[ \text{A.3.1 Differentiation of a Scalar with Respect to a Vector} \]

The derivative of a scalar with respect to a vector is the vector whose components are the derivatives of the scalar with respect to each of the components of the vector. If \( \mathbf{r} = (x, y, z)^T \), then
\[ \frac{df}{d\mathbf{r}} = (f_x, f_y, f_z)^T. \]
Consequently,
\[ \frac{d}{da} (\mathbf{a} \cdot \mathbf{b}) = \mathbf{b} \quad \text{and} \quad \frac{d}{db} (\mathbf{a} \cdot \mathbf{b}) = \mathbf{a}. \]
The length of a vector is the square root of the sum of the squares of its elements,
\[ |\mathbf{a}| = \sqrt{\mathbf{a} \cdot \mathbf{a}}. \]
We see that
\[ \frac{d}{da} |\mathbf{a}|^2 = 2\mathbf{a}, \]
so that
\[ \frac{d}{da} |\mathbf{a}| = \hat{\mathbf{a}}, \]
where \( \hat{\mathbf{a}} = \mathbf{a}/|\mathbf{a}| \); also,
\[ \frac{d}{da} (\mathbf{a} \cdot \mathbf{b} \cdot \mathbf{c}) = \mathbf{b} \times \mathbf{c}, \]
where \( [\mathbf{a} \cdot \mathbf{b} \cdot \mathbf{c}] = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) \) as before. Furthermore,
\[ \frac{d}{d\mathbf{a}} \mathbf{a}^T \mathbf{M} \mathbf{b} = \mathbf{M} \mathbf{b} \quad \text{and} \quad \frac{d}{d\mathbf{b}} \mathbf{a}^T \mathbf{M} \mathbf{b} = \mathbf{M}^T \mathbf{a}. \]
In particular,
\[ \frac{d}{d\mathbf{x}} \mathbf{x}^T \mathbf{M} \mathbf{x} = (\mathbf{M} + \mathbf{M}^T) \mathbf{x}. \]
The derivative of a scalar with respect to a matrix is the matrix whose components are the derivatives of the scalar with respect to the elements of the matrix. Thus if
\[ \mathbf{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \]
then
\[ \frac{df}{d\mathbf{M}} = \begin{pmatrix} \frac{df}{da} & \frac{df}{db} \\ \frac{df}{dc} & \frac{df}{dd} \end{pmatrix}. \]
Consequently,
\[ \frac{d}{d\mathbf{M}} \text{Trace}(\mathbf{M}) = \mathbf{I}, \]
where the trace of a matrix is the sum of its diagonal elements and \( \mathbf{I} \) is the identity matrix. Also,
\[ \frac{d}{d\mathbf{M}} \mathbf{a}^T \mathbf{M} \mathbf{b} = \mathbf{a} \mathbf{b}^T. \]
In particular,
\[ \frac{d}{d\mathbf{M}} \mathbf{x}^T \mathbf{M} \mathbf{x} = \mathbf{x} \mathbf{x}^T. \]
Note that \( \mathbf{a} \mathbf{b}^T \) is not the scalar \( \mathbf{a} \cdot \mathbf{b} \). The latter equals \( \mathbf{a}^T \mathbf{b} \). If \( \mathbf{a} = (a_x, a_y, a_z)^T \) and \( \mathbf{b} = (b_x, b_y, b_z)^T \), the dyadic product is
\[ \mathbf{a} \mathbf{b}^T = \begin{pmatrix} a_x b_x & a_x b_y & a_x b_z \\ a_y b_x & a_y b_y & a_y b_z \\ a_z b_x & a_z b_y & a_z b_z \end{pmatrix}. \]
Another interesting matrix derivative is
\[ \frac{d}{d\mathbf{M}} \text{Det}(\mathbf{M}) = \text{Det}(\mathbf{M}) (\mathbf{M}^{-1})^T. \]