Further Results on Sensor Network Localization Using Rigidity

Tolga Eren, Walter Whiteley, and Peter N. Belhumeur

***Abstract*— Two further results, which extend the previous work on the use of rigidity in sensor network localization, are given. The previous work provided the conditions for the localization of an entire network in which some nodes know their locations and other nodes determine their locations by measuring the distances to their neighbors. First, the paper gives the conditions for partial localization of a subnetwork when an entire network is not localizable. Second, the paper gives the conditions for localization in which some nodes know their locations and other nodes determine their locations by measuring the bearings (angle of arrivals) to their neighbors rather than the distances.**

I. INTRODUCTION

Rigidity theory has been used in network localization recently [1]–[5]. In [2], necessary and sufficient conditions were given for the localization of an entire network in which some nodes know their locations and other nodes determine their locations by measuring the distances to their neighbors. This paper extends the previous work in two ways. First, partially localization is explored when it is not possible to localize all the nodes in an entire network. Consider the sensor network shown in Figure 1. It can be verified that the conditions for unique localization given in [2] are not satisfied for this network. However, it will be shown that some nodes in such a network can still be localized.

Second, the use of rigidity in the localization problem for nodes that use bearing information is considered. In the general model of sensor networks, there are usually some nodes named *beacons*, whose position information is known. Those beacons have either GPS or they are manually configured. For the rest of the nodes there are two types of node capabilities: (i) distance measurements (also called ranging) to their neighbors, and (ii) bearing measurements (also called angle of arrival (AOA)) between their neighbors. In [2], sensor nodes that measure distances are considered. Here, the use of rigidity is extended for networks in which nodes use bearing information. A *bearing* is the angle between a sensing/communication link and the x-axis of a node's local coordinate system [6]. For example, if two nodes i and j have a sensing/communication link between each other as shown in Figure 2, then *bearing* information for i and j, denoted by θ_{ij} and θ_{ji} respectively, are the angles between the x-axis of each node's local coordinate system and the sensing/communication link (i, j). If each node uses its own coordinate system and is not aware of other nodes' coordinate systems, then nodes will not be able to reach a consensus to make use of the bearing information. In real implementations of bearing information, the information about a global coordinate system (x_G, y_G) is either known by all nodes using a compass or is transmitted from beacons to ordinary nodes [6]. This is done by passing "heading" information from one node to another. By *heading* is meant the angle between the y-axis of the global coordinate system and the x-axis of the node's local coordinate system. For example, ϕ_i is the heading of i in Figure 3. Once node i passes the information ϕ_i and θ_{ij} to node j, then node j can compute its heading by $\phi_j = \pi - (\theta_{ij} - \phi_i) + \theta_{ji}$.

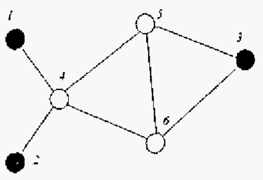

Fig. 1. A simple example of a sensor network. Filled circles denote beacons, and empty circles denote ordinary nodes.

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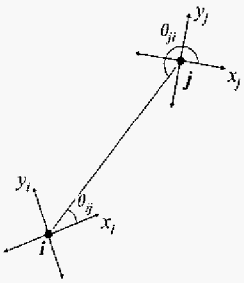

Fig. 2. Bearing information for i and j, denoted by θ_{ij} and θ_{ji} respectively, are the angles between the x-axis of each node's local coordinate system and the sensing/communication link (i, j).

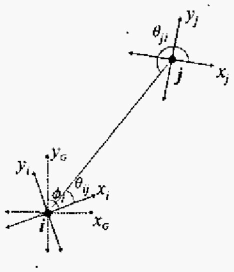

Fig. 3. ϕ_i is the heading of i.

Once nodes know the global coordinate system, they can transform the bearing information measured in their local coordinate systems (θ_{ij} and θ_{ji}) into bearing information in the global coordinate system (Θ_{ij} and Θ_{ji}) as shown in Figure 4. We note that $\Theta_{ji} = \pi + \Theta_{ij}$.

The rest of this paper is organized as follows. A brief review of the formulation of the network localization problem presented in [2] is given in §II. Partial localization in subnetworks is considered in §III. In Section §IV, we study rigidity for sensor networks with bearing information.

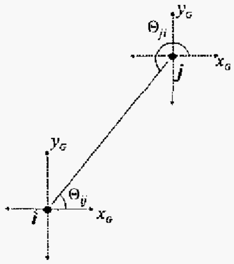

Fig. 4. Bearing information in the global coordinate system are Θ_{ij} and Θ_{ji}.

II. REVIEW OF THE FORMULATION OF THE NETWORK LOCALIZATION PROBLEM

The "network localization problem with distance information" can be formulated as follows. One begins with a network $\mathbf{N}$ in real 2-dimensional space consisting of a set of $m > 0$ nodes labelled 1 through m which represent "beacons" together with $n - m > 0$ additional nodes labelled $m+1$ through n which represent sensors. Each node is located at a fixed position in $\mathbb{R}^2$ and has associated with it a specific set of "neighboring" nodes. $\mathbf{N}$'s neighbor relationships can be conveniently described by an undirected graph $\mathbb{G}_{\mathbf{N}} = \{\mathcal{V}, \mathcal{E}_{\mathbf{N}}\}$ with vertex set $\mathcal{V} = \{1, 2, \ldots, n\}$ and edge set $\mathcal{E}_{\mathbf{N}}$ defined so that (i, j) is one of the graph's edges just in case nodes i and j are neighbors. We assume throughout, that $\mathbb{G}_{\mathbf{N}}$ is a connected graph. The *network localization problem with distance information* is to determine the locations x_i of all sensor nodes in $\mathbb{R}^2$ given the graph of the network $\mathbb{G}_{\mathbf{N}}$, the positions of the beacons x_j, $j \in \{1, 2, \ldots, m\}$ in $\mathbb{R}^d$, and the distance $\delta_{\mathbf{N}}(i, j)$ between each neighbor pair $(i, j) \in \mathcal{E}_{\mathbf{N}}$. The network localization problem just formulated is said to be *solvable* if there is exactly one set of vectors $\{x_{m+1}, \ldots x_n\}$ in $\mathbb{R}^2$ which is consistent with the given data $\mathbb{G}_{\mathbf{N}}$, $\{x_1, x_2, \ldots, x_m\}$, and $\delta_{\mathbf{N}} : \mathcal{E}_{\mathbf{N}} \to \mathbb{R}$. . By a 2-dimensional *point formation* at $p \triangleq$ column $\{p_1, p_2, \cdots, p_n\}$, written $\mathbb{F}_p$, is meant a set of n points $\{p_1, p_2, \cdots, p_n\}$ in $\mathbb{R}^2$ together with a set $\mathcal{L}$ of k *links*, labelled (i, j), where i and j are distinct integers in $\{1, 2, \ldots, n\}$; the *length* of link (i, j) is the Euclidean distance between point p_i and p_j. In this context, the points p_i represent the positions of nodes {i.e., both sensors and beacons} in $\mathbb{R}^2$ and the links in $\mathcal{L}$ label those specific node pairs whose inter-node distances are given. Thus for the sensor network discussed above, $\mathcal{L}$ would consist of not only all pairs in $\mathcal{E}_{\mathbf{N}}$, but also all additional beacon pairs (i, j), $i, j \in \{1, 2, \ldots, m\}$ since the distances between pairs of beacons are uniquely specified by their position vectors which are given. The resulting graph with those extra edges is called *grounded graph* and is denoted by $\hat{\mathbb{G}}_{\mathbf{N}} = \{\mathcal{V}, \mathcal{L}\}$.

Two point formations $\mathbb{F}_p$ and $\mathbb{F}_q$ are *congruent* if they have the same graph and if q and p are congruent. It is clear that $\mathbb{F}_p$ is uniquely determined by its graph and distance function *at most* up to a congruence transformation. A formation which is *exactly* determined up to congruence by its graph and distance function is called *globally rigid*. In [2], it is shown that a formation is globally rigid if the grounded graph is generically globally rigid. For a detailed discussion of rigidity and global rigidity, see for example [2], [7]. A *cut set* is a minimal set of vertices that, when deleted, disconnects the graph. For example, $\{3,4\}$ is a cut set of the underlying graph $\mathbb{G}_{\mathrm{N}}$ of the sensor network shown in Figure 1. The connectivity of a graph is the number of vertices in its minimum cut set. There is the following test for global rigidity of a graph:

Theorem 2.1: (Jackson-Jordán [8]): A graph $\mathbb{G}$ with $n \geq 4$ vertices is generically globally rigid in $\mathbb{R}^2$ if and only if it is 3-connected and redundantly rigid in $\mathbb{R}^2$.

III. PARTIAL LOCALIZATION IN SUBNETWORKS

Although $\hat{\mathbb{G}}_{\mathrm{N}}$ fails the conditions given in Theorem 2.1, it might still be possible to localize some of the nodes in the network. A straightforward way is as follows: Assume that the underlying grounded graph of the network $\hat{\mathbb{G}}_{\mathrm{N}} = (\mathcal{V}, \mathcal{L})$, does not satisfy the conditions of Theorem 2.1. But let us assume that there exists subgraph(s) of $\hat{\mathbb{G}}_{\mathrm{N}}$, namely $\hat{\mathbb{G}}^1_{\mathrm{N}}, \hat{\mathbb{G}}^2_{\mathrm{N}}, \ldots, \hat{\mathbb{G}}^k_{\mathrm{N}}$ that satisfy the conditions of Theorem 2.1. Then all the nodes in the subnetworks $\mathrm{N}_1, \mathrm{N}_2, \ldots, \mathrm{N}_k$ (with the underlying grounded graphs $\hat{\mathbb{G}}^1_{\mathrm{N}}, \hat{\mathbb{G}}^2_{\mathrm{N}}, \ldots, \hat{\mathbb{G}}^k_{\mathrm{N}}$ respectively) are localizable. Next, we consider a more challenging case.

A. Implicitly Extended Grounded Graph

Let us assume that a grounded graph $\hat{\mathbb{G}}_{\mathrm{N}}$ is redundantly rigid and 2-connected rather than 3-connected. Hence it does not satisfy the conditions of Theorem 2.1. Assume that there exists a cut set $\mathcal{C} = \{v_1, v_2\}$ where v_1, v_2 are cut vertices. Consider the two graph components $\mathbb{G}_1 = (\mathcal{V}_1, \mathcal{L}_1)$, $\mathbb{G}_2 = (\mathcal{V}_2, \mathcal{L}_2)$ that share only these two cut vertices (and possibly the edge (v_1, v_2)), and whose union is the entire grounded graph, i.e., $\mathbb{G}_1 \cup \mathbb{G}_2 = \hat{\mathbb{G}}_{\mathrm{N}}$ and $\mathbb{G}_1 \cap \mathbb{G}_2 = \{v_1, v_2\}$. An edge is called an *implicit edge* if its presence or absence does not change the rigidity of a graph. Let $\hat{\mathbb{G}}^*_{\mathrm{N}}$ be the grounded graph obtained by inserting the implicit edge (v_1, v_2) to $\hat{\mathbb{G}}_{\mathrm{N}}$. (If (v_1, v_2) already exists, we keep it as it is.) $\hat{\mathbb{G}}^*_{\mathrm{N}}$ is called *implicitly extended grounded graph*. Let us consider each component $\mathbb{G}_1$ and $\mathbb{G}_2$ together with this implicit edge and denote them by $\hat{\mathbb{G}}_1 = (\mathcal{V}_1, \mathcal{L}_1 \cup \{(v_1, v_2)\})$ and $\hat{\mathbb{G}}_2 = (\mathcal{V}_2, \mathcal{L}_2 \cup \{(v_1, v_2)\})$. We have the following proposition:

Proposition 3.1: If $\hat{\mathbb{G}}_2$ is rigid and $\hat{\mathbb{G}}_1$ is 3-connected and redundantly rigid, and $\hat{\mathbb{G}}_1$ has at least three beacons then all the vertices of $\hat{\mathbb{G}}_1$ are localizable.

Proof: Recall that an implicit edge does not change the rigidity properties of a graph. We are given that $\hat{\mathbb{G}}_2$ is rigid, and the edge (v_1, v_2) is an implicit edge. Hence we can consider the entire network with this implicit edge inserted, and the rigidity property of the network remains the same. After inserting (v_1, v_2), since $\hat{\mathbb{G}}_1$ is redundantly rigid and 3-connected, then all the vertices in $\hat{\mathbb{G}}_1$ are localizable. ■

For example, consider the sensor network shown in Figure 1. The grounded graph of this network, shown in Figure 5, does not satisfy the conditions in Theorem 2.1. Therefore it is not possible to localize all the nodes $\{4,5,6\}$ in this network. As explained before, $\{3,4\}$ is the cut set of this graph. Note that the subgraph $\mathbb{G}_2 = (\mathcal{V}_2, \mathcal{L}_2)$ where $\mathcal{V}_2 = \{3,4,5,6\}$ and $\mathcal{L}_2 = \{(3,5),(3,6),(4,5),(4,6),(5,6)\}$ is rigid and hence the edge $(3,4)$ is an implicit edge. We insert this implicit edge to the grounded graph as shown in Figure 6. Note that the subgraph $\mathbb{G}_1 = (\mathcal{V}_1, \mathcal{L}_1 \cup \{(3,4)\})$ where $\mathcal{V}_1 = \{1,2,3,4\}$ and $\mathcal{L}_1 = \{(1,2),(2,3),(1,3),(1,4),(2,4)\}$ satisfies the conditions in Theorem 2.1, which means that all the nodes in $\mathbb{G}_1$ are localizable. Therefore node 4 can be localized.

IV. POINT FORMATIONS BASED ON BEARINGS

The *network localization problem with bearing information* can be formulated in a similar way as in §II. The only difference is that instead of having the distance $\delta_{\mathrm{N}}(i,j)$ between each neighbor pair $(i,j) \in \mathcal{E}_{\mathrm{N}}$, we now have bearings $\beta_{\mathrm{N}}(i,j)$ between each neighbor pair $(i,j) \in \mathcal{E}_{\mathrm{N}}$. Note that there are two bearing information for each edge measured by the two end-nodes of each edge. The network localization problem is said to be *solvable* if there is exactly one set of vectors $\{x_{m+1}, \ldots x_n\}$ in $\mathbb{R}^2$ which is consistent with

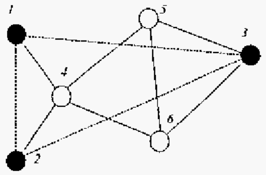

Fig. 5. The grounded graph of the network shown in Figure 1.

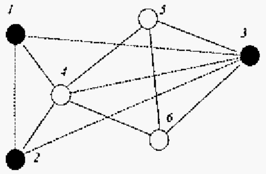

Fig. 6. The grounded graph shown with the additional inserted implicit edge $(3,4)$.

the given data $\mathbb{G}_{\mathrm{N}}$, $\{x_1, x_2, \ldots, x_m\}$, and $\beta_{\mathrm{N}} : \mathcal{V}_{\mathcal{E}_{\mathrm{N}}} \rightarrow [0, 2\pi)$ where $\mathcal{V}_{\mathcal{E}_{\mathrm{N}}}$ denote the set of the end-vertices of the edges in $\mathcal{E}_{\mathrm{N}}$.

Two point formations on the same graph are *parallel drawings* if corresponding edges are parallel [7]. Given a fixed point formation $\mathbb{F}_p$, we are interested in parallel drawings $\mathbb{F}_q$ in which $q_i - q_j$ is parallel to $p_i - p_j$ for all $(i, j) \in \mathcal{E}_{\mathrm{N}}$. Using the operator $(.)^{\perp}$, for turning a plane vector by $\frac{\pi}{2}$ counterclockwise, these constraints can be written:

$$(p_i - p_j)^{\perp} \cdot (q_i - q_j) = 0. \tag{1}$$

Each such constraint is called a *direction* constraint in CAD literature. This gives a system of $|\mathcal{E}_{\mathrm{N}}|$ homogeneous linear equations. A solution of this system is called a *parallel point formation.* Trivially parallel point formations are translations and dilations of the original point formation, including the parallel point formation in which all points are coincident. All others are non-trivial. A point formation with bearing constraints is called *parallel rigid* if all parallel point formations are trivially parallel. Otherwise it is called *flexible.* Taking the derivative of (1) (recall that p is a fixed point set and $q(t)$ is time varying in (1)), we obtain

$$(p_i - p_j)^{\perp} \cdot (\dot{q}_i(t) - \dot{q}_j(t)) = 0, \qquad (i, j) \in \mathcal{E}_{\mathrm{N}}, \quad t \geq 0 \tag{2}$$

These equations can be rewritten in matrix form as

$$R_{\mathcal{B}}(p)\dot{q} = 0 \tag{3}$$

where $\dot{q}$ = column $\{\dot{q}_1, \dot{q}_2, \ldots, \dot{q}_n\}$ and $R_{\mathcal{B}}(p)$ is the rigidity matrix for formations with bearing information. An n-point formation is parallel rigid if rank $R_{\mathcal{B}} = 2n - 3$.

Example 4.1: Consider a planar point formation $\mathbb{F}_p$ with bearing constraints shown in Figure 7. We assume that at least one node (a beacon) knows the global coordinate system and the information about this global coordinate system is passed to the other nodes in the formation. The same point formation drawn with bearing constraints in the global coordinate system is shown in Figure 8. This has a rigidity matrix as shown in Table II. We note that given two points $p_i = (x_i, y_i)$ and $p_j = (x_j, y_j)$, then $(p_i - p_j)^{\perp} = (y_i - y_j, x_j - x_i)$.

It is shown in [9] that any statement for a point formation of distances can be given for the same point formation of directions where distances are switched with directions. The isomorphism goes down the pairs of columns for each vertex, turning all the vectors by 90° (in a direction of choice). This process preserves the solution space (just turning the solutions by 90° as well), and turns each row for a distance into a row for a direction. Because of this geometric switching, there is a generic switching theorem in [9] that converts results in a direct fashion. Thus the generic type of rigidity is defined in the same manner as in the case of distances given in [2]. The graph theoretic test is given with the following theorem:

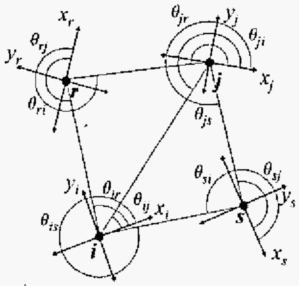

Fig. 7. A point formation with bearing constraints.

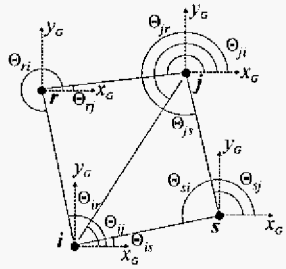

Fig. 8. The same point formation drawn with bearing constraints in the global coordinate system.

Theorem 4.2: A graph $\mathbb{G} = (\mathcal{V}, \mathcal{E}_{\mathrm{N}})$ is generically parallel rigid in 2-dimensional space if and only if there is a subset $\mathcal{E}'_{\mathrm{N}} \subseteq \mathcal{E}_{\mathrm{N}}$ satisfying the following two conditions: (1) $|\mathcal{E}'_{\mathrm{N}}| = 2|\mathcal{V}| - 3$, (2) For all $\mathcal{E}''_{\mathrm{N}} \subseteq \mathcal{E}'_{\mathrm{N}}, \mathcal{E}''_{\mathrm{N}} \neq \emptyset, |\mathcal{E}''_{\mathrm{N}}| \leq 2|\mathcal{V}(\mathcal{E}''_{\mathrm{N}})| - 3$, where $|\mathcal{V}(\mathcal{E}''_{\mathrm{N}})|$ is the number of vertices that are end-vertices of the edges in $\mathcal{E}''_{\mathrm{N}}$.

A. *Rigidity Implies Global Rigidity for Networks with Bearing Information*

Although global rigidity implies rigidity for networks with distance information, the reverse is not true. Therefore the conditions for global rigidity is much stronger than rigidity for networks with distance information. On the other hand, as we prove below, for

$R_{\mathcal{B}}(p)$	i		j		r		s	
(i,j)	$y_i - y_j$	$x_j - x_i$	$y_j - y_i$	$x_i - x_j$	0	0	0	0
(i,r)	$y_i - y_r$	$x_r - x_i$	0	0	$y_r - y_i$	$x_i - x_r$	0	0
(i,s)	$y_i - y_s$	$x_s - x_i$	0	0	0	0	$y_s - y_i$	$x_i - x_s$
(j,r)	0	0	$y_j - y_r$	$x_r - x_j$	$y_r - y_j$	$x_j - x_r$	0	0
(j,s)	0	0	$y_j - y_s$	$x_s - x_j$	0	0	$y_s - y_j$	$x_j - x_s$

TABLE I

RIGIDITY MATRIX EXAMPLE FOR BEARINGS

networks with bearing information rigidity also implies global rigidity. We have the following theorem:

Theorem 4.3: If $\mathbb{F}_p$ and $\mathbb{F}_q$ are parallel formations in 2-dimensional space, then $\mathbb{F}_p$ is rigid if and only if $\mathbb{F}_p$ is globally rigid under similarity maps.

Proof: Suppose that $\mathbb{F}_p$ is not globally rigid. Therefore, there is a parallel drawing $\mathbb{F}_q$ which is not similar to $\mathbb{F}_p$ as a configuration. We will show that $\mathbb{F}_p$ is flexible with $\mathbb{F}_q$ as a non-trivial parallel drawing. For all edges $(i,j) \in \mathcal{B}$, $(p_i - p_j)$ is parallel to $(q_i - q_j)$. Therefore, $(p_i - p_j)^{\perp} \cdot (q_i - q_j) = 0$ as required. Since $\mathbb{F}_p$ is not similar to $\mathbb{F}_q$, there is some pair $(h,k) \notin \mathcal{B}$ such that $p_h - p_k$ is not parallel to $q_h - q_k$. Therefore, $(p_h - p_k)^{\perp} \cdot (q_h - q_k) \neq 0$. This confirms that $\mathbb{F}_q$ is a non-trivial parallel drawing of $\mathbb{F}_p$. Conversely, suppose that $\mathbb{F}_p$ is flexible with a non-trivial parallel drawing $\mathbb{F}_q$. Then $\mathbb{F}_q$ itself is the non-similar parallel drawing of $\mathbb{F}_p$ which shows it is not globally rigid. ■

Note that at least one distance information is needed to avoid dilation in point formations with bearing information. This can be easily achieved by using two beacons in the network since the distance between two beacons is implicitly known.

B. Bilaterations

It is possible to derive useful sufficient conditions and sequential constructions for generically parallel rigid networks with bearing information in a similar way that trilaterations are used for generically globally rigid networks with distance information [2]. *Bilateration* is the operation whereby a node with known distances to two other nodes determines its own position in terms of the positions of those two neighbors. Starting with two beacons, all other nodes can determine their nodes sequentially, which is also computationally efficient to realize. In rigidity theory, the related graph expansion operation is called the *vertex addition*: given a minimally rigid graph $\mathbb{G} = (\mathcal{V}, \mathcal{L})$, we add a new vertex i with two edges between i and two other vertices in $\mathcal{V}$.

Theorem 4.4: (bilateration [7]) Let $\mathbb{G}^* = (\mathcal{V}^*, \mathcal{L}^*)$ be a graph with a vertex i of degree 2 in 2-dimensional space; let $\mathbb{G} = (\mathcal{V}, \mathcal{L})$ denote the subgraph obtained by deleting i and the edges incident with it. Then $\mathbb{G}$ is minimally parallel rigid if and only if $\mathbb{G}^*$ is minimally parallel rigid.

A longer version of this paper is available as a technical report [10].

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