A tableau method for graded intersections of modalities: a case for concept languages

Ani Nenkova (`ani@cs.columbia.edu`)
Columbia University, Department of Computer Science

Abstract. A concept language with role intersection and number restriction is defined and its modal equivalent is provided. The main reasoning tasks of satisfiability and subsumption checking are formulated in terms of modal logic and an algorithm for their solution is provided. An axiomatization for a restricted graded modal language with intersection of modalities (the modal counterpart of the concept language we examine) is given and used in the proposed algorithm.

Keywords: graded modalities, axiomatization, proof theory

1. Motivation

This paper is mainly concerned with investigating the question of which new features show up in a modal language when counting and intersection of modalities are introduced. The focus is on finding axioms describing the interaction between counting and intersection. To our knowledge, no such axiomatization has been published so far. There are two reasons for choosing to work with a restricted modal language (without negation and disjunction) rather than with the full modal language. First, by doing so we can concentrate on the new features of the language that are due to counting and intersection. Second, the approach of syntactic decomposition of formulas in the process of finding the axiomatization also gives a way for subsumption checking in a concept language without negation where the usual approach (checking for satisfiability of a concept and the negation of another concept) cannot be taken.

2. From concept languages to modal logic

Concept languages are knowledge representation languages devised for expressing knowledge about concepts and concept hierarchies. The basic building blocks are primitive concepts and primitive roles. Concepts describe the common properties of a collection of individuals and are interpreted as the sets of individuals possessing those properties. Roles are interpreted as binary relations between individuals. In each concept language a number of language constructs (such as intersection, union, role quantification, etc.) are offered and used for the construction of compound concepts and roles. The main reasoning tasks are satisfiability and subsumption checking and in most cases other deductive tasks can be reduced to those two. The central role of subsumption in concept language reasoning has motivated numerous studies of its computational complexity (e.g. (Donini et al, 1994), (Donini et al, 1991)) but in the work to follow we will not be interested in the efficiency of computation or in possible implementations and optimization details.

A concept language is defined in which concepts (denoted by C and D) can be built according to the following syntactic rule:

$C, D \longrightarrow \top \mid \bot \mid A \mid \neg A \mid C \bigcap D \mid \forall R.C \mid (\geq n\ R) \mid (\leq n\ R)$,
where A is a primitive concept, n is a natural number and R is a primitive role or an intersection of primitive roles.

A pair $\langle W, I\rangle$ is called an *interpretation* for the concept language if W is a non-empty set, I is a function, mapping every concept from the language to a subset of W and every role to a subset of $W \times W$, and

$I(\top) = W$, $I(\bot) = \emptyset$
$I(\neg A) = W \setminus I(A)$
$I(C \cap D) = I(C) \cap I(D)$
$I(\forall R.C) = \{a \in W \mid \forall b((a, b) \in I(R) \Rightarrow b \in I(C))\}$
$I((\geq \mathrm{n\ R}))= \{a \in W \mid| \{b \mid (a, b) \in I(R)\} |\geq n\}$
$I((\leq \mathrm{n\ R}))= \{a \in W \mid| \{b \mid (a, b) \in I(R)\} |\leq n\}$
$I(R \cap Q) = I(R) \cap I(Q)$, where R and Q are roles.

An interpretation $\langle W, I\rangle$ is called a model for a concept C if $I(C)$ is a non-empty set. A concept is said to be satisfiable if a model for it exists and unsatisfiable otherwise. C is subsumed by D if $I(C) \subseteq I(D)$ for every interpretation.

Definition. A modal language is defined, consisting of:
- constant symbols $\top$ and $\bot$;
- a countable set of propositional variables $P = \{p_1, p_2, p_3, \ldots\}$;
- conjunction & and negation $\neg$;
- Op—a set of unary operators of the kind $[R]$;
- Oq— a set of unary operators of the kind $[R]_{\mathrm{n}}$ and $\langle R\rangle_{\mathrm{n}}$, where n is a natural number.

Definition. A formula in the modal language is inductively defined by the formation rules:
- $\top$ and $\bot$ are formulas;
- every propositional variable is a formula;
- if p is a propositional variable, then $\neg p$ is a formula;
- if $\langle R\rangle_{\mathrm{n}}$, $[R]_{\mathrm{n}} \in Oq$, then $\langle R\rangle_{\mathrm{n}}\top$ and $[R]_{\mathrm{n}}\bot$ are formulas;
- if φ and ψ are formulas, then $\varphi\&\psi$ is a formula;

- if φ is a formula and $[R] \in Op$, then $[R]\varphi$ is a formula.

Definition. $\mathcal{M} = \langle M, \mathcal{R}, v\rangle$, where M is a non-empty set, for every element in Op and Oq of the kind $[R], \langle R\rangle_n, [R]_n$ there is a corresponding element R in $\mathcal{R}$ such that $R \subseteq M \times M$ and v is a valuation ($v : M \times P \to \{0,1\}$, where P is the set of variables in the language) is called a model for the modal language.

Definition. $\mathcal{M} = \langle M, \mathcal{R}, v\rangle$ is tree-like if $\langle M, \bigcup\limits_{R\in\mathcal{R}} R\rangle$ is a tree.

Definition. The truth conditions for a formula φ in a model $\mathcal{M} = \langle M, \mathcal{R}, v\rangle$ and $x \in M$ are inductively defined by:

i) $\mathcal{M}, x \models \top$ and $\mathcal{M}, x \not\models \bot$ for all $x \in M$.
ii) $\mathcal{M}, x \models p$ iff $v(x,p) = 1$;
iii) $\mathcal{M}, x \models \neg p$ iff $v(x,p) = 0$;
iv) $\mathcal{M}, x \models \varphi_1 \& \varphi_2$ iff $\mathcal{M}, x \models \varphi_1$ and $\mathcal{M}, x \models \varphi_2$;
v) $\mathcal{M}, x \models [R]\varphi$ iff $\forall y(xRy \Rightarrow \mathcal{M}, y \models \varphi)$ for $[R] \in Op$;
vi) $\mathcal{M}, x \models \langle R\rangle_n \top$ iff $|\{y \mid (x,y) \in R\}| > n$ for $\langle R\rangle_n \in Oq$.
vii) $\mathcal{M}, x \models [R]_n \bot$ iff $|\{y \mid (x,y) \in R\}| \leq n$ for $[R]_n \in Oq$.

According to the above definition $[R]\bot$ is equivalent to $[R]_0\bot$ and occasionally we will switch from one notation to the other.

There is a direct correspondence between the described concept language and the language of basic modal logic. More exactly, transformations δ and Δ exist, such that δ translates concepts into modal formulas and Δ transforms models for a concept to models for the modal formula corresponding to that concept. The connection between modal and concept languages was first considered in (Schild, 1990).

For the transformations δ and Δ we have:

$\delta(\top) = \top$, $\delta(\bot) = \bot$

$\delta(A_i) = p_i$ and $\delta(\neg A_i) = \neg p_i$, where A_i is a primitive concept, and p_i—a propositional variable.

$\delta(C \cap D) = \delta(C) \& \delta(D)$

$\delta(\forall R.C) = [R]\delta(C)$

$\delta(\geq$ n R$) = \langle R\rangle_{n-1}\top$

$\delta(\leq$ n R$) = [R]_n\bot$, and

$\Delta(\langle M, I\rangle) = (M, \{R\}_{R\in\mathcal{R}}, v)$, where $\mathcal{R}$ is the set of binary relation symbols, corresponding to the roles in the concept language, and $v(x,p) = 1$ iff $x \in I(A)$ and $\delta(A) = p$.

In fact, $x \in I(C) \Leftrightarrow \Delta(\langle M, I\rangle), x \models \delta(C)$.

Definition. Let Γ be a set of modal formulas, $\mathcal{M}$—a model and $x \in M$. We will write $\mathcal{M}, x \models \Gamma$, if $\mathcal{M}, x \models \varphi$ for every $\varphi \in \Gamma$.

Definition. In order to express the problems of satisfiability and subsumption from the concept language, we introduce configurations of the kind $\Gamma \mapsto \psi$, where $\Gamma \cup \psi$ is a finite set of modal formulas. Informally, $\Gamma \mapsto \psi$ means that in every model $\mathcal{M}$ and $x \in M$ ψ is true, given that all the formulas from Γ are true. Formally,

$\Gamma \mapsto \psi \; \Omega \; \forall \mathcal{M} \forall x \in M (\mathcal{M}, x \models \Gamma \Rightarrow \mathcal{M}, x \models \psi)$

The configuration $\Gamma \mapsto \psi$ will be verbalized as "ψ follows from Γ" or "Γ implies ψ", or we will also say "the configuration is admissible". If Γ is a singleton, $\Gamma = \{\varphi\}$, we will directly write $\varphi \mapsto \psi$.

By writing $\mathcal{M}, x \not\models \Gamma \mapsto \psi$ we denote that the model $\mathcal{M}$ and the point x in it witness that the configuration $\Gamma \mapsto \psi$ is not admissible, that is, that ψ does not follow from Γ.

The reason for introducing the above configurations is the lack of implication (disjunction and negation) in the examined modal language.

Now it can be said that a concept C is satisfiable iff the configuration $\delta(C) \mapsto \perp$ is not admissible, that is—a model M and a point x in it exist such that $\mathcal{M}, x \not\models \delta(C) \mapsto \perp$. Similarly, C subsumes D iff $\delta(D)$ follows from $\delta(C)$. Also, C does not subsume D iff a model $\mathcal{M}$ and a point x in it exist such that $\mathcal{M}, x \not\models \delta(C) \mapsto \delta(D)$.

From here on the entailment problem $\varphi \mapsto \psi$ will be examined. Section 3 deals with this problem in a modal language with three modalities and their intersections. This case is general enough and shows the way the solution can be formulated for a language with arbitrary number of modalities and their intersections. The existence of an algorithm for solving the entailment problems and the properties of this algorithm show that the satisfiability and subsumption tasks are decidable. The algorithms for solving the entailment problem $\varphi \mapsto \psi$ also play the role of completeness theorems for the formal system introduced in the next section.

3. Three Modalities and Their Intersections

We are interested in recognizing the admissibility of the configuration $\varphi \mapsto \psi$. More exactly, we are looking for an algorithm, that given a configuration $\varphi \mapsto \psi$ returns a witness for its inadmissibility or a formal proof for it. The witness for inadmissibility is a tree-like model and its root. The work to follow can be considered an ideological continuation of (Tinchev, 1993). The traditional tableau method proof techniques from (Fitting, 1983; Smullyan, 1968) are used—the two formulas in the configuration are systematically decomposed and in the process of decomposition the truth conditions for a formula are exploited in order to build a model in which the left-hand side of the configuration is true and the right-hand side is not. Obstacles for building such a model are used as indications about how a formal proof for the configuration is to be obtained.

The formulas we will work with are of the form
$\varphi ::= p \mid \neg p \mid \perp \mid \top \mid \varphi_1 \& \varphi_2 \mid [R]\varphi \mid \langle R\rangle_{\mathrm{n}}\top \mid [R]_{\mathrm{n}}\perp$.

There are three basic modalities in the language (R_1, R_2, R_3) and the remaining modalities are interpreted in the models as the set theoretic intersection of the interpretations of the basic ones and respectively denoted as $R_{12} = R_1 \cap R_2$, R_{13}, R_{23}, R_{123}. For convenience, we will sometimes write $[a], [a]_{\mathrm{n}}, \langle a\rangle_{\mathrm{n}}$ instead of $[R_{\mathrm{a}}], [R_{\mathrm{a}}]_{\mathrm{n}}$ and $\langle R_{\mathrm{a}}\rangle_{\mathrm{n}}$. Also, we will say that index i is part of index j if the interpretation of R_{j} is the result of intersecting the interpretation of R_{i} with some other relation from the interpretation of the language (that is, if we look at the indices as numeric strings, all digits that appear in i appear in j as well).

The structural properties and axioms $Ax1$ to $Ax13$ in the formal system we will define are taken from (Van der Hoek et al., 1992).

1. $\mapsto$ has the following structural properties-rules of inference:
   Monotonicity $\Gamma \mapsto \psi \Rightarrow \Gamma \bigcup \{\phi\} \mapsto \psi$
   Cut $\Gamma \mapsto \psi$ and $\Gamma \bigcup \{\psi\} \mapsto \chi \Rightarrow \Gamma \mapsto \chi$
   *Distribution** $\Gamma \mapsto \psi \Rightarrow \{[R]\gamma \mid \gamma \in \Gamma\} \mapsto [R]\psi$

2. The configuration-axioms are:
   Ax1 $p, \neg p \mapsto \perp$
   Ax2 $\varphi \mapsto \top$
   Ax3 $\varphi, \psi \mapsto \varphi \& \psi$
   Ax4 $\varphi \& \psi \mapsto \varphi$ and $\varphi \& \psi \mapsto \psi$
   Ax5 $[R](\varphi \& \psi) \mapsto ([R]\varphi \& [R]\psi)$ for every R
   Ax6 $[R]\varphi, [R]\psi \mapsto [R](\varphi \& \psi)$ for every R
   Ax7 $[R]\perp, \langle R\rangle\top \mapsto \perp$ for every R
   Ax8 $\perp \mapsto \varphi$
   Ax9 $\langle i\rangle_{\mathrm{n}}\top \mapsto \langle i\rangle_{\mathrm{n}-1}\top$ for every i
   Ax10 $[i]_{\mathrm{n}}\perp \mapsto [i]_{\mathrm{n}+1}\perp$ for every i
   Ax11 $\langle i\rangle_{\mathrm{n}}\top \mapsto \langle j\rangle_{\mathrm{n}}\top$, if the index j is part of the index i
   Ax12 $[i]_{\mathrm{n}}\perp \mapsto [j]_{\mathrm{n}}\perp$, if the index i is part of the index j
   Ax13 $[i]_{\mathrm{n}}\perp \& \langle i\rangle_{\mathrm{n}}\top \mapsto \perp$

   Unfortunately, the combination of intersection and counting has deeper consequences than those handled by the above axioms. Given a model $\mathcal{M}$ and a point x in M, let x_{i} be the number of all R_{i}-successors of x and $X_1 = \{(x, y) \mid xR_1y\}$, $X_2 = \{(x, y) \mid xR_2y\}$, $X_3 = \{(x, y) \mid xR_3y\}$, $X_{12} = X_1 \cap X_2$, $X_{13} = X_1 \cap X_3$, $X_{23} = X_2 \cap X_3$ and $X_{123} = X_1 \cap X_2 \cap X_3$.

$Ax12$ and $Ax13$ capture the idea that, for example, $X_{123} \subseteq X_{12} \subseteq X_1$ and $X_{123} \subseteq X_{13} \subseteq X_1$, but those are seen as independent facts and we don't have any way to express the requirement that $X_{12} \cup X_{13}$ is a subset of X_1 and consequently $|X_{12} \cup X_{13}| \leq |X_1|$, and thus $x_{12} + x_{13} - x_{123} \leq x_1$.

By similar reasoning one can see that in order to satisfy the requirement some of the relations to be the set theoretic intersection of others, conditions in the form of the above inequality should be fulfilled for the number of successors of point x at which we want a formula with graded modal part to true.

The axiom that follows incorporates those conditions.

Ax14 Let γ be the formula that consists of graded modal part only, with at most one modal operator per relation, and each $[i]$ has index m_i and each $\langle i \rangle$ has index n_i.

Then $\gamma \mapsto \perp$, if there is no solution $\langle x_1, x_2, x_3, x_{12}, x_{13}, x_{23}, x_{123} \rangle$ to the following system of inequalities A:

$n_k < x_k \leq m_k$ for $k = 1, 2, 3$
$n_{12} < x_{12} \leq min\{m_1, m_2, m_{12}\}$
$n_{13} < x_{13} \leq min\{m_1, m_3, m_{13}\}$
$n_{23} < x_{23} \leq min\{m_2, m_3, m_{23}\}$
$n_{123} < x_{123} \leq min\{m_1, m_2, m_3, m_{12}, m_{13}, m_{23}, m_{123}\}$
$x_i \leq x_j$ for all pairs of i and j such that j is part of i.
$x_1 - x_{12} - x_{13} + x_{123} \geq 0$
$x_2 - x_{12} - x_{23} + x_{123} \geq 0$
$x_3 - x_{13} - x_{23} + x_{123} \geq 0$

Remark. The first inequalities rule out the possibility of obtaining evidence for the insatisfiability of γ by application of axioms $Ax1$ to $Ax13$. The inequality $x_1 + x_2 + x_3 - x_{12} - x_{13} - x_{23} + x_{123} \geq 0$ is not included in A because it follows from the rest.

Remark. Axiom $Ax14$ gives necessary and sufficient conditions for a formula γ to be made true at some point x. If the system A has a solution, the required number of successors can be built so that γ becomes true. First, x_{123} R_1-, R_2- and R_3- (R_{123}-)successors of x are added. Then, new $x_{12} - x_{123}$ R_1- and R_2- (R_{12}-), new $x_{13} - x_{123}$ R_1- and R_3- (R_{13}-) and new $x_{23} - x_{123}$ R_2- and R_3- (R_{23}-) successors of x are added. Finally, new $x_1 - x_{12} - x_{13} + x_{123}$ R_1-, $x_2 - x_{12} - x_{23} + x_{123}$ R_2- and $x_3 - x_{13} - x_{23} + x_{123}$ R_3-successors of x are added. The possibility to build the required number of new successors at each step is guaranteed by the fact that $\overline{x}$ is a solution to the system A.

Ax15 $\gamma \mapsto \langle i \rangle_{\mathrm{p}} \top$, if for every solution to the system A we have that $p < x_{\mathrm{i}}$

Ax16 $\gamma \mapsto [i]_{\mathrm{q}} \bot$, if for every solution to the system A we have that $x_{\mathrm{i}} \leq q$

Definition. A literal is a propositional variable or its negation.

Definition. The number of nested modal operators in a configuration or a formula φ is called modal depth of the configuration or φ and is denoted by $d(\varphi \mapsto \psi)$ or $d(\varphi)$ respectively:
- $d(L) = 0$, L—literal or constant;
- $d(\varphi \& \psi) = max\{d(\varphi), d(\psi)\}$;
- $d(O\psi) = d(\psi) + 1$, if $O \in Op \cup Oq$ and ψ is a formula;
- $d(\varphi \mapsto \psi) = max\{d(\varphi), d(\psi)\}$.

Definition. A sequence $\Gamma_1 \mapsto \psi_1$, $\Gamma_2 \mapsto \psi_2$,..., $\Gamma_{\mathrm{k}} \mapsto \psi_{\mathrm{k}} = \Gamma \mapsto \psi$ is called a formal proof for $\Gamma \mapsto \psi$ if all of its terms are either axioms or are obtained by preceding configurations by the rules of inference.

Property 1. If $\chi \mapsto \varphi$ and $\varphi \mapsto \psi$, then $\chi \mapsto \psi$ (*Transitivity* of $\mapsto$).

Property 2. $\{\varphi_1, ..., \varphi_{\mathrm{n}}\} \mapsto \psi \Leftrightarrow \varphi_1 \& ... \& \varphi_{\mathrm{n}} \mapsto \psi$

Property 3. $\varphi_1 \mapsto \psi_1$, $\varphi_2 \mapsto \psi_2 \Rightarrow \varphi_1 \& \varphi_2 \mapsto \psi_1 \& \psi_2$.

Correctness Theorem. If a formal proof for $\Gamma \mapsto \psi$ exists, then this configuration is admissible.

Example 1. Let us have a formula that consists of graded modal part only. For better legibility, the formula is represented in a table–the first row shows the diamond indices and the second row the box indices for the corresponding relation symbol:

R_1	R_2	R_3	R_{12}	R_{13}	R_{23}	R_{123}	
2	3	4	2	1	0	0	3
3	6	8	3	3	6	1	2

A model for this formula cannot be built. The tuple $\langle 3, 4, 5, 3, 2, 1, 1 \rangle$ cannot be a solution to the system A for this formula because $x_1 - x_{12} - x_{13} + x_{123} = -1$ and none of the positive terms can grow because of the restrictions imposed by the box operators in the formula.

Example 2. Let us have a formula with the following graded modal part:

R_1	R_2	R_3	R_{12}	R_{13}	R_{23}	R_{123}	
0	3	4	2	1	0	0	3
8	6	8	3	3	6	2	2

There is a solution to the system A for this formula—$\langle 3, 4, 5, 3, 2, 2, 2\rangle$ and x_1 cannot be less than 3, even though there is no explicit requirement for such number of successors in the formula itself.

Theorem. An algorithm $\mathcal{A}$ exists such that for any configuration $\varphi \mapsto \psi$ after a finite number of steps $\mathcal{A}$ returns a tree-like witness model or a proof for $\varphi \mapsto \psi$.

Proof.

Since the potential conjunctive terms in the formulas from the configuration can be too numerous, the following notations are introduced:

- the greatest index of a $\langle R_i\rangle$ term in φ is denoted by n_i;
- the least index of a $[R_i]$ term in φ is denoted by m_i;
- the greatest index of a $\langle R_i\rangle$ term in ψ is denoted by $\overline{n}_i$;
- the least index of a $[R_i]$ term in ψ is denoted by $\overline{m}_i$;

All the other indices can be ignored.

- B_φ , B_ψ — the boolean part of φ and ψ respectively;
- M_φ , M_ψ —the non-graded modal part of φ and ψ respectively;
- G_φ , G_ψ —the graded modal part of φ and ψ respectively;

Definition. If ψ_1 is one of the conjunctive terms in ψ, we will write $\psi_1 ; \psi$.

Base: If $d(\varphi \mapsto \psi) = 0$, the witness model or the proof is found by direct propositional arguments.

Inductive assumption: Let the algorithm be already built for $d(\varphi \mapsto \psi) = n$.

Let the modal depth of $\varphi \mapsto \psi$ be $n + 1$.

A witness model will not exist if φ is itself unsatisfiable. Cases 1 to 5 cover all the possible ways in which φ can be unsatisfiable and show a proof for the initial configuration. Otherwise we proceed by attempting to extend one possible model for φ so that φ remains true but ψ is false in it. If all such attempts fail, we end up with a proof for the initial configuration.

1. If $\perp$ occurs in φ, we have the following proof: $\varphi \mapsto \perp$, $\perp \mapsto \psi$ and by transitivity we obtain $\varphi \mapsto \psi$.

2. If p and $\neg p$ occur in the boolean part of φ, we have the proof $\varphi \mapsto p\&\neg p$, $p\&\neg p \mapsto \perp$, $\perp \mapsto \psi$ and by transitivity we obtain $\varphi \mapsto \psi$.

3. Formula φ is unsatisfiable because its graded modal part is unsatisfiable.

   If $\langle i\rangle_{\mathrm{n}}\top$ and $[j]_{\mathrm{m}}\perp$ occur in φ, and $n \geq m$ and the index j is part of the index i, we have $\langle i\rangle_{\mathrm{n}}\top \mapsto \langle j\rangle_{\mathrm{n}}\top$, $\langle j\rangle_{\mathrm{n}}\top \mapsto \langle j\rangle_{\mathrm{m}}\top$, but $\langle j\rangle_{\mathrm{m}}\top$, $[j]_{\mathrm{m}}\perp \mapsto \perp$. Thus we obtain a proof for $\varphi \mapsto \psi$.

   If none of the above is the case, we have the following possibilities:

   The valuation is defined at a point x in a way, such that $x \models L_1$ for all L_1 ; B' .

   Definition. A sub-problem engendered by a relation R_{i} from the configuration $\varphi \mapsto \psi$ is the configuration that should be examined if B' and B have shown that φ does not necessarily imply ψ but there is a R_{i}-successor requirement in φ ($\langle R_{\mathrm{i}}\rangle_{\mathrm{n}_i}\top$; φ), so further examinations are needed. That is the configuration $\varphi_1 \mapsto \perp$ where $\varphi_1 = \bigwedge_{\mathrm{j\ part\ of\ i}}\{\bigwedge\{N|[j]N\ ;\ \varphi\}\}$.

   The algorithm is set to work on the sub-problems engendered by the relations R_{i} for which $\langle i\rangle_{\mathrm{n}_i}\top$ occurs in φ. All of those subproblems are of modal depth less or equal to n and therefore for all the sub-problems either a witness model $\mathcal{M}_{\mathrm{i}}$ or a proof will be found. Any time a prove is returned by the algorithm, it will be extended to a prove for the initial configuration $\varphi \mapsto \psi$.

4. Formula φ is unsatisfiable because it has a term that requires a successor to be made, but the non-graded modal part of φ is unsatisfiable.

   Let $\langle i\rangle_{\mathrm{n}_i}\top$ occur in φ, but the algorithm returns a proof for the subproblem $M \mapsto \perp$ engendered by R_{i}. This proof can be continued in the following way: $[i]M \mapsto [i]\perp$, $\varphi \mapsto [i]M$, $\varphi \mapsto [i]\perp$, $\varphi \mapsto \langle i\rangle_{\mathrm{n}_i}\top$, $\langle i\rangle_{\mathrm{n}_i}\top \mapsto \langle i\rangle\top$, $[i]\perp\&\langle i\rangle\top \mapsto \perp$, $\varphi \mapsto \perp$, $\perp \mapsto \psi$, $\varphi \mapsto \psi$.

5. Assume that there is no solution to the system A for the graded modal part of φ. Then $\varphi \mapsto \perp$, $\perp \mapsto \psi$, $\varphi \mapsto \psi$.

6. Formula φ does not have successor requirements and falsifying ψ can be achieved by boolean arguments, so a trivial witness model is built.

If (*) $\exists L$-literal$(L\ ;\ B_{\text{-}}\ \&L\ ;\!/\ B_{\text{'}}\ )$ and no $\langle R\rangle_{\mathrm{n}}$ occurs in φ, then $\langle\{x\},\{R_1 = \emptyset, R_2 = \emptyset, R_3 = \emptyset\}, v\rangle$ is a witness model for the configuration, where $v(L,x) = 1$ for all L occurring in $B_{\text{'}}$ and $v(L,x) = 0$ for some L that satisfies (*).

For all the cases to follow we assume that $v(L,x) = 1$ for all L occurring in $B_{\text{'}}$.

7. Formula φ has successor requirements and ψ can be made false either by boolean arguments or the first solution to A happened to violate some of its successor conditions. Later, if nothing else succeeds we will try all the solutions for A.

   Let $\overline{x}$ be a solution to the system A for the graded modal part of φ and some $\langle i\rangle_{\mathrm{n}_i}\top$ occurs in φ and
   a) (*) $\exists L$-literal$(L\ ;\ B_{\text{-}}\ \&L\ ;\!/B_{\text{'}}\ )$, or
   b) for some i $\overline{m}_{\mathrm{i}} < x_{\mathrm{i}}$ or $x_{\mathrm{i}} \leq \overline{n}_{\mathrm{i}}$, or

   In case a), the valuation is extended with $v(L,x) = 0$ for some L satisfying (*).

   In both 7a and 7b $x \not\models \psi$. To build a witness model, for every i we add y_{i} copies of $\mathcal{M}_{\mathrm{i}}$ as successors to x, where $\langle x_1, x_2, x_3, x_{12}, x_{13}, x_{23}, x_{123}\rangle$ is a solution to the system A for φ, $\mathcal{M}_{\mathrm{i}}$ is the witness model for the sub-problem engendered by R_{i}, and $y_{123} = x_{123}$, $y_{12} = x_{12} - x_{123}$, $y_{13} = x_{13} - x_{123}$, $y_{23} = x_{23} - x_{123}$, $y_1 = x_1 - x_{12} - x_{13} + x_{123}$, $y_2 = x_2 - x_{12} - x_{23} + x_{123}$, $y_3 = x_3 - x_{13} - x_{23} + x_{123}$.

Definition. A complete sub-problem engendered by the relation R_{i} from the configuration $\varphi \mapsto \psi$ is the configuration $\varphi_1 \mapsto \psi_1$ where $\varphi_1 = \bigwedge_{\mathrm{j\ part\ of\ i}}\{\bigwedge\{N|[j]N\ ;\ \varphi\}\}$ and $\psi_1 = \bigwedge_{\mathrm{j\ part\ of\ i}}\{\bigwedge\{S|[j]S\ ;\ \psi\}\}$. In other words, it is the configuration that should be explored if the examinations so far show that φ implies ψ but $d(\varphi \mapsto \psi) > 0$ and R_{i}-successor can be made at the next step.

8. Now we try to falsify ψ by looking into its non-graded boolean part.

   The sub-problems for all the relations are considered. Their modal depth is less or equal to n and therefore for all R_{i} the algorithm will return either a witness model $\mathcal{N}_{\mathrm{i}}$ or a proof for the complete sub-problem engendered by R_{i}.

   a) If a suitable witness model for the subproblem is found, it is extended to a witness model for the initial configuration.

If the algorithm returns a witness model for at least one of the complete sub-problems and a solution $\overline{z}$ for the system A exists such that for the relation R_i, engendering the complete sub-problem, $z_i > 0$, then the witness model $\mathcal{M}$ for the initial configuration is build as in 6, but in the y_i successors of x copies of $\mathcal{N}_i$ are added instead of copies of $\mathcal{M}_i$. Obviously, $\mathcal{N}_i \not\models N_i$ and therefore $\mathcal{M}, x \not\models \psi$

b) Even if we found witness models for some sub-problems, we might be unable to use them because φ does not allow for the building of any more successors and then a proof for the initial configuration is obtained.

If $z_i = 0$ for every solution $\overline{z}$ to the system A, then $[j]\perp$ occurs in φ for some index j that is part of the index i. In this case we have a proof for $\varphi \mapsto N_i$—$\perp \mapsto N_i$, $[i]\perp \mapsto [i]N_i$, $\varphi \mapsto [j]\perp$, $[j]\perp \mapsto [i]\perp$, $\varphi \mapsto [i]N_i$.

9. If 1–8 are not the case and (**) $\forall i(\overline{n}_i \leq n_i \& m_i \leq \overline{m}_i)$, then $\varphi \mapsto \psi$ because $G_\varphi \mapsto G_\psi$ from (**), $M_\varphi \mapsto M_\psi$ from 8 (all complete subproblems returned with a proof and those are easily extended by *Distribution**), $B_\varphi \mapsto B_\psi$ from 7.

10. Finally, all the possibilities to break a successor requirement in ψ are systematically explored. If any of them succeeds—a witness model is obtained, else we end up with a proof of the fact that ψ follows form φ.

    If (#) $\exists i(n_i < \overline{n}_i < m_i)$ or (##) $\exists i(n_i < \overline{m}_i < m_i)$, we examine the formulas $\varphi_{\overline{n}_i}$ that is the same as φ except that the index of $[i]$ is $\overline{n}_i$ instead of m_i if i satisfies (#), and $\varphi_{\overline{m}_i}$ that is the same as φ except that the index of $\langle i \rangle$ is $\overline{m}_i$ instead of n_i if i satisfies (##).

    For all i satisfying (#) or (##), the system A for the formula described above is formed. If there is a solution to A for some of those formulas, a witness model for the initial configuration is build as in 7.

    If there is no solution to the system A for any of the modified formulas, then for every solution $\overline{x}$ to the system for φ $x_i > \overline{n}_i$ and $x_i \leq \overline{m}_i$. If this were not so, some of the systems for the modified formulas would have had a solution.

    So, we have that $G_\varphi \mapsto G_\psi$, but $M_\varphi \mapsto M_\psi$ from 7 and $B_\varphi \mapsto B_\psi$ from 6 and therefore we have a proof for $\varphi \mapsto \psi$.

4. Conclusions

The formal system introduced in section three shows what the axiomatization of the restricted graded modal language with intersection of arbitrary number modalities should look like. Set-theoretic principals are incorporated in the axioms and they indicate that the problem of satisfying the graded modal part of a formula at a point can in fact be transformed into the problem of realizing a tuple of numbers as mutually intersecting sets (showing that sets with cardinalities specified in the tuple can indeed exist). Working with a restricted language allows us to exemplify more clearly these aspects that become somewhat obscured by technical complications in the case of a full modal language.

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