

Problem Proposal
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The solution to this problem will be used in a user-interface for editing 1D functions.

Find a closed-form expression for a monotonic 1D function that connects two points with known slope. For a more hands-on explanation, the criteria here seem to coincide with the way a strip of paper interpolates two points with known slope – given that you allow the strip's length to grow or shrink as it desires.

Here are some examples of good and bad solutions:

Good:



Bad:



Find $y = f(x)$ such that:

1. $f(x_a) = y_a$
2. $f(x_b) = y_b$
3. $f'(x_a) = m_a$
4. $f'(x_b) = m_b$
5. $f'(x)$ exists, $x \in (x_a, x_b)$

1-5 indicate that the function passes through the points (x_a, y_a) and (x_b, y_b) , and has the desired slopes m_a and m_b at those points, and is C1. (These requirements can be fulfilled with a cubic hermite spline.)

6. $x_a < x_b$
7. $y_a < y_b$
8. $0 \leq m_a$
9. $0 \leq m_b$

6-9 do not impose any new limitations on the solution, but are necessary for the criteria that will follow.

10. $0 \leq f'(x), x \in [x_a, x_b]$

10 indicates that this is a non-decreasing function. Work has been done on monotone interpolation [Fritsch and Carlson 1980; Huynh 1993], but this does not satisfy the next requirements.

$$11. f'(x_i) = 0 \wedge f'(x_j) = 0 \wedge x_i, x_j \in (x_a, x_b) \Rightarrow x_i = x_j$$

$$12. f''(x_i) = 0 \wedge f''(x_j) = 0 \wedge x_i, x_j \in (x_a, x_b) \Rightarrow x_i = x_j$$

11 and 12 are my attempt to state mathematically that there should not be staircases, and there should not be straight segments, respectively. If you have a more elegant/correct way of stating this, feel free to use it.

I think another way to state this in a single statement is:

$$11.2 f'(x_i) = f'(x_j) = f'(x_k) \wedge x_i, x_j, x_k \in (x_a, x_b) \Rightarrow x_i = x_j \vee x_i = x_k \vee x_j = x_k$$

In other words, the same slope can occur up to twice in the given interval.

There is a caveat to the above rules – the straight line solution. In some cases, a straight line connecting the two end points can fulfill 1-10. In that case, 11-12 should not be imposed. Again, if you find a more elegant way of stating 11-12 that allows for the straight-line solution without making it an exception, you're probably better off using that.

13. As a final condition, the solution should be continuous in the sense that infinitesimally small changes in $x_a, y_a, x_b, y_b, m_a, m_b$ lead to infinitesimally small changes in $f(x)$. As one set of inputs approaches another set of inputs (from any direction), so should the solution of the first approach the solution of the second.

If this is interesting to you, or you would like to discuss it further, please email me at aner@cs.columbia.edu

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PS. It is possible that a solution to 1-13 may exist, but that it is degenerate in some sense, and therefore not useful for my specific application (though this is not likely). However, even such a solution would probably be of use elsewhere. To help guide the process think of the strip of paper. Let it slide between your fingers, and adjust the tension so that it gives a pleasing interpolant.

Fritsch, F.N.; Carlson, R.E. Monotone Piecewise Cubic Interpolation. SIAM Journal on Numerical Analysis, Vol. 17, No. 2. (Apr., 1980), pp. 238-246.

Huynh, Hung T. 1993. Accurate Monotone Cubic Interpolation. SIAM Journal on Numerical Analysis, Vol. 30, No. 1. (Feb., 1993), pp. 57-100