1 Inverse Kinematics

1. Forward Kinematics is a mapping from joint space $Q$ to Cartesian space $W$:

$$ F(Q) = W $$

This mapping is one to one - there is a unique Cartesian configuration for the robot for a given set of joint variables. Inverse Kinematics is a method to find the inverse mapping from $W$ to $Q$:

$$ Q = F^{-1}(W) $$

2. The inverse kinematics problem has a wide range of applications in robotics. Most of our high level problem solving about the physical world is posed in Cartesian space. While we can reason about the physical world in Cartesian terms, the robot is actuated in joint space - that is what we ultimately can control. Once we solve a problem for its Cartesian space constraints, we need to map these constraints into the robot’s joint space using inverse kinematics. For example, if we specify a straight line trajectory for a robot arm, we need to break that trajectory into a set of joint space values over time to get the robot to follow the line.

3. The inverse kinematics mapping is typically one to many. There are usually multiple sets of joint variables that will yield a particular Cartesian configuration. When solving the inverse problem, we often have to choose one solution from a number of valid solutions. There are also degenerate cases with an infinite number of solutions (called singularities).

4. Some solutions of the inverse mapping may not be physically realizable. This is due to manipulators having physical joint limits that prevent the mechanism from achieving certain joint configurations that may be solutions to the inverse kinematics problem (e.g. a joint may not have a full 360 degree motion).

5. There may not be a closed form solution to the inverse problem at all for some manipulators. However, most manipulators use a 3 DOF wrist that has intersecting axes. This allows us to separate the inverse problem into a 3 DOF problem for finding the endpoint of the wrist and a 3 DOF problem for finding the orientation of the wrist. This does in fact have a closed form solution.

6. Numerical methods can be used to find a solution to the inverse problem if a closed form solution does not exist.

7. A redundant robot is one that has extra DOF’s (more than the space the robot works in requires). For example, a 7-DOF robot has an extra DOF if it is used in our normal 6-DOF Cartesian space. This can be useful for reaching around obstacles, and avoiding collisions with other objects in the workspace.
8. To solve inverse kinematics, we use a variety of methods: geometric, trigonometric and algebraic. There are certain forms that you can recognize and then use the appropriate method to solve for a joint variable.

9. Once you solve for a joint variable, you can think of the manipulator as a reduced DOF mechanism - with one less joint. Now solve this manipulator's inverse problem and keep doing this until all joints are solved for.

2 Cylindrical and Spherical Robot Inverse Kinematics

![Diagram of a 3-DOF Spherical Robot](image1)

![Diagram of a 3-DOF Cylindrical Robot](image2)

Figure 1: Top: 3-DOF Spherical Robot. This robot's final transform is found by $\text{Rot}(Z, \theta) \text{Rot}(Y, \phi) \text{Trans}(x, r)$ (note: these are not D-H transforms). Bottom: 3-DOF Cylindrical Robot. This robot's final transform is found by $\text{Rot}(Z, \theta) \text{Trans}(Z, d) \text{Trans}(x, r)$ (note: these are not D-H transforms)
1. **CYLINDRICAL:** Given the final $T$ matrix of the cylindrical robot with $P = (P_x, P_y, P_z)$, find $\Theta, d, r$ for the cylindrical robot.

$$T_3^0 = \begin{bmatrix}
\cos \Theta & -\sin \Theta & 0 & r \cos \Theta \\
\sin \Theta & \cos \Theta & 0 & r \sin \Theta \\
0 & 0 & 1 & d \\
0 & 0 & 0 & 1
\end{bmatrix}$$

2. First, we look for positions in the $T$ matrix that have a single variable isolated. $P_z$ in the matrix $T_3^0$ (last column, third row) is only dependent on the variable $d$, so we can directly solve for $d$:

$$d = P_z.$$

3. We notice that $P_x^2 + P_y^2 = r^2$. So $r = \pm \sqrt{P_x^2 + P_y^2}$.

4. To solve for $\Theta$, we can take the ratio of $P_y$ and $P_x$:

$$\frac{P_y}{P_x} = \frac{r \sin \Theta}{r \cos \Theta} = \tan \Theta, \quad \Theta = \arctan(P_y, P_x)$$

5. Note there are two solutions for $r$ and $\Theta$ values to reach this position in space. If we take the positive value of $r$, then $\Theta = \arctan2(P_y, P_x)$, and if we take the negative value of $r$ then $\Theta = \arctan2(-P_y, -P_x)$

The two solutions are equivalent to $(r, \Theta)$ and $(-r, \Theta + 180)$. However, the negative value of $r$ may not be physically realizable in an actual robot - the arm may only extend radially forward. Also, consider $P = (0, 0, K)$ - can we solve for inverse kinematics in this robot configuration?

**SPHERICAL** robot has 3 variables: Radius $r$, Longitude angle $\Theta$ and latitude angle $\Phi$. For the Spherical robot the final matrix is:

$$T_3^0 = \begin{bmatrix}
\cos \Theta \cos \Phi & -\sin \Theta & \cos \Theta \sin \Phi & r \cos \Phi \cos \Theta \\
\sin \Theta \cos \Phi & \cos \Theta & \sin \Theta \sin \Phi & r \cos \Phi \sin \Theta \\
-\sin \Phi & 0 & \cos \Phi & -r \sin \Phi \\
0 & 0 & 0 & 1
\end{bmatrix}$$

1. $r$ is the length of the the spherical arm radius and is equal to:

$$r = \pm \sqrt{P_x^2 + P_y^2 + P_z^2}$$

2. To solve for $\Phi$, $P_z = -r \sin \Phi$, and $\Phi = \sin^{-1}(P_z / -r)$. Note $r$ has 2 values from above.

3. As in the cylindrical robot, the ratio of $P_x$ and $P_y$ yields a solution for $\Theta$:

$$\frac{P_y}{P_x} = \frac{r \cos \Phi \sin \Theta}{r \cos \Phi \cos \Theta} = \tan \Theta, \quad \Theta = \arctan2(P_y, P_x) \text{ or } \Theta = \arctan2(-P_y, -P_x)$$

4. As in cylindrical robot, there are multiple solutions. To get the second solution, find the point on the sphere directly opposite where the manipulator is (if at $P_x, P_y, P_z$ choose $-P_x, -P_y, -P_z$). The second solution is the latitude and longitude of the new point with $r$’s value negated. It also may not be physically realizable with the manipulator. Note singularity $P = (0, 0, K)$. 

3
3 Adept Robot Inverse Kinematics

Figure 2: Adept 1 Scara Robot arm. This arm is in a R-R-P-R configuration. \( \theta_1, \theta_2, \theta_4 \) are the revolute joint angle variables and \( q_3 \) is the prismatic joint variable. The robot is pictured in the Home position in the frame diagram using the values of the joint variables listed in the table below.

1. Given the final position of the robot \( P=(P_x, P_y, P_z) \). Find \( \Theta_1, \Theta_2, q_3, \) and \( \Theta_4 \) for the scara robot. The final \( T \) matrix is given below:

\[
T^0_4 = \begin{bmatrix}
C_{1-2-4} & S_{1-2-4} & 0 & a_1C_1 + a_2C_{1-2} \\
S_{1-2-4} & -C_{1-2-4} & 0 & a_1S_1 + a_2S_{1-2} \\
0 & 0 & -1 & d_1 - q_3 - d_4 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

2. To find \( \Theta_2 \): if we square and sum \( P_x \) and \( P_y \), we can get an expression in \( \Theta_2 \):

\[
P_x^2 + P_y^2 = (a_1C_1 + a_2C_{1-2})^2 + (a_1S_1 + a_2S_{1-2})^2
\]

\[
P_x^2 + P_y^2 = a_1^2 + a_2^2 + 2a_1a_2C_1(C_1C_2 + S_1S_2) + 2a_1a_2S_1(S_1C_2 - C_2S_1)
\]

\[
P_x^2 + P_y^2 = a_1^2 + a_2^2 + 2a_1a_2C_1^2C_2 + 2a_1a_2S_1^2C_2
\]
Figure 3: Solution to $\Theta_2$ of Adept, as seen from above (along Z axis)

\[ P_x + P_y^2 = a_1^2 + a_2^2 + 2a_1a_2C_2 \]

\[ C_2 = \frac{P_x^2 + P_y^2 - a_1^2 - a_2^2}{2a_1a_2} \]

\[ S_2 = \sqrt{1 - C_2^2} \]

\[ \Theta_2 = \pm \cos^{-1} \left( \frac{P_x^2 + P_y^2 - a_1^2 - a_2^2}{2a_1a_2} \right) \]

3. This is really just the derivation of the Law of Cosines which we can also use to find $\Theta_2$ (see figure above):

\[ a_1^2 + a_2^2 - 2a_1a_2\cos(180 - \Theta_2) = P_x^2 + P_y^2 \quad (\text{Law of Cosines}) \]

\[ \cos(180 - \Theta_2) = \frac{P_x^2 + P_y^2 - a_1^2 - a_2^2}{-2a_1a_2} \]

\[ -\cos(\Theta_2) = \frac{P_x^2 + P_y^2 - a_1^2 - a_2^2}{-2a_1a_2} \]

\[ \cos(\Theta_2) = \frac{P_x^2 + P_y^2 - a_1^2 - a_2^2}{2a_1a_2} \]
4. To solve for $\Theta_1$, we solve for the following:

\[ a_1C_1 + a_2C_{1-2} = P_x \quad \text{Two equations in two unknowns} \ (C_1, S_1) \]
\[ a_1S_1 + a_2S_{1-2} = P_y \quad \text{(} \Theta_2 \text{ known from above)} \]

\[ a_1C_1 + a_2C_1C_2 + a_2S_1S_2 = P_x \ , \ a_1S_1 + a_2S_1C_2 - a_2S_2C_1 = P_y \]

\[ (a_1 + a_2C_2)C_1 + (a_2S_2)S_1 = P_x \ , \ (-a_2S_2)C_1 + (a_1 + a_2C_2)S_1 = P_y \]

\[ S_1 = \frac{a_2S_2P_x + (a_1 + a_2C_2)P_y}{(a_2S_2)^2 + (a_1 + a_2C_2)^2} \]

\[ C_1 = \frac{(a_1 + a_2C_2)P_x - a_2S_2P_y}{(a_2S_2)^2 + (a_1 + a_2C_2)^2} \]

\[ \Theta_1 = \atan_2(a_2S_2P_x + (a_1 + a_2C_2)P_y, (a_1 + a_2C_2)P_x - a_2S_2P_y) \]

5. To solve for $q_3$:

\[ P_z = d_1 - q_3 - d_4 \ ; \ q_3 = d_1 - d_4 - P_z \]

6. To solve for $\Theta_4$: The final roll angle cannot be determined from the position vector $\{P_x, P_y, P_z\}$. If we are given the orientation matrix, then we can use the ratios of $N_x, N_y$ to find $\Theta_4$.

\[ \tan_{1-2-4} = \frac{S_{1-2-4}}{C_{1-2-4}} = \frac{N_y}{N_x} \]

\[ \Theta_1 - \Theta_2 - \Theta_4 = \atan_2(N_y, N_x) \]

\[ \Theta_4 = -\atan_2(N_y, N_x) + \Theta_1 - \Theta_2 \]
7. Example Solution (no $\Theta_4$) if $a_1 = \sqrt{3}$, $a_2 = 1$, $d_1 = 5$, $d_4 = 2$ and $P = [\sqrt{3},-1,1]$, solve for joint variables:

$$\Theta_2 = \pm \cos^{-1} \left( \frac{P_x^2 + P_y^2 - a_1^2 - a_2^2}{2a_1a_2} \right) = \pm \cos^{-1} \left( \frac{0}{2\sqrt{3}} \right) = \pm 90^\circ$$

$$\Theta_1 = \arctan \left( \frac{a_2 P_x}{a_1 + a_2 C_2} \right) \arctan \left( \frac{a_1 + a_2 C_2}{a_2 P_y - a_2 S_2 P_y} \right)$$

- if $\Theta_2 = +90^\circ$, $\Theta_1 = \arctan(0,4) = 0^\circ$
- if $\Theta_2 = -90^\circ$, $\Theta_1 = \arctan(-\sqrt{3},1) = -60^\circ$

$$q_3 = d_1 - d_4 - P_z = 5 - 2 - 1 = 2$$

Two Solutions:

<table>
<thead>
<tr>
<th>$\Theta_1$</th>
<th>$\Theta_2$</th>
<th>$q_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0$^\circ$</td>
<td>90$^\circ$</td>
<td>2</td>
</tr>
<tr>
<td>-60$^\circ$</td>
<td>-90$^\circ$</td>
<td>2</td>
</tr>
</tbody>
</table>