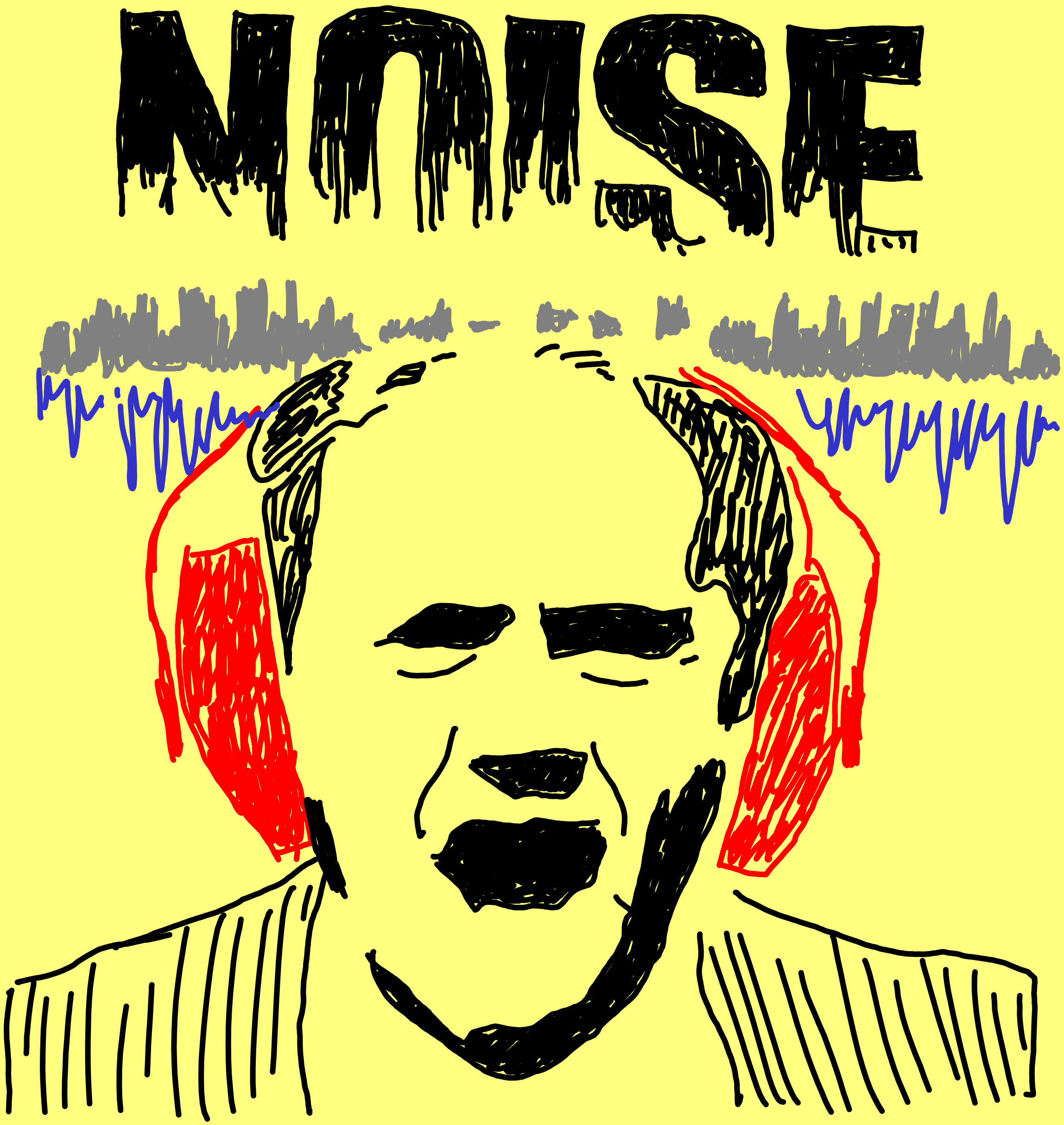
Measuring Scale before Simplification

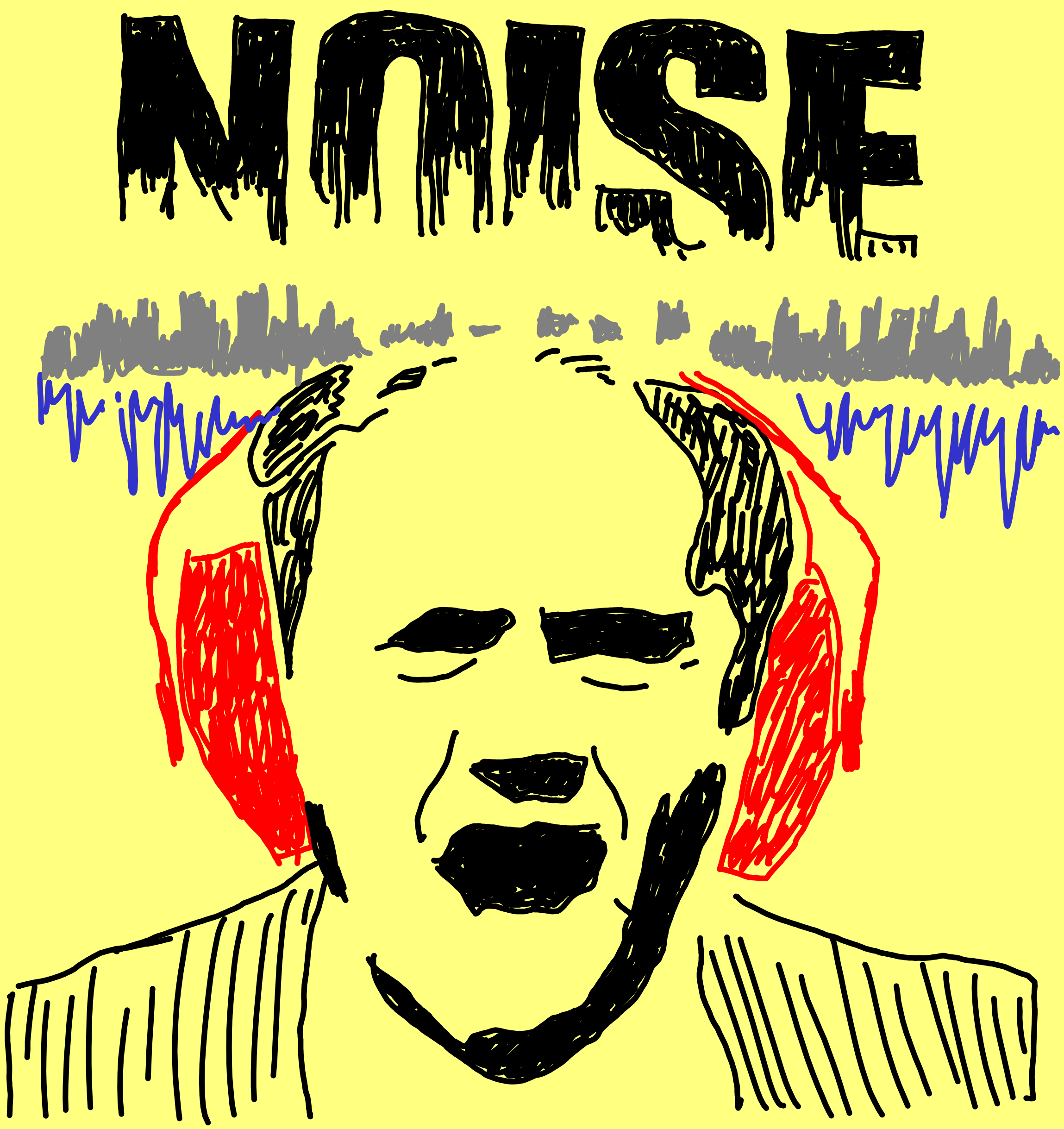
Columbia

Measuring Scale before Simplification

Herbert Edelsbrunner

Duke University
& Geomagic


NOISE



. . . IS ALWAYS AND EVERYWHERE.

Measure

Measure

. . . and don't simplify unless you must.

I Persistence

II Curves

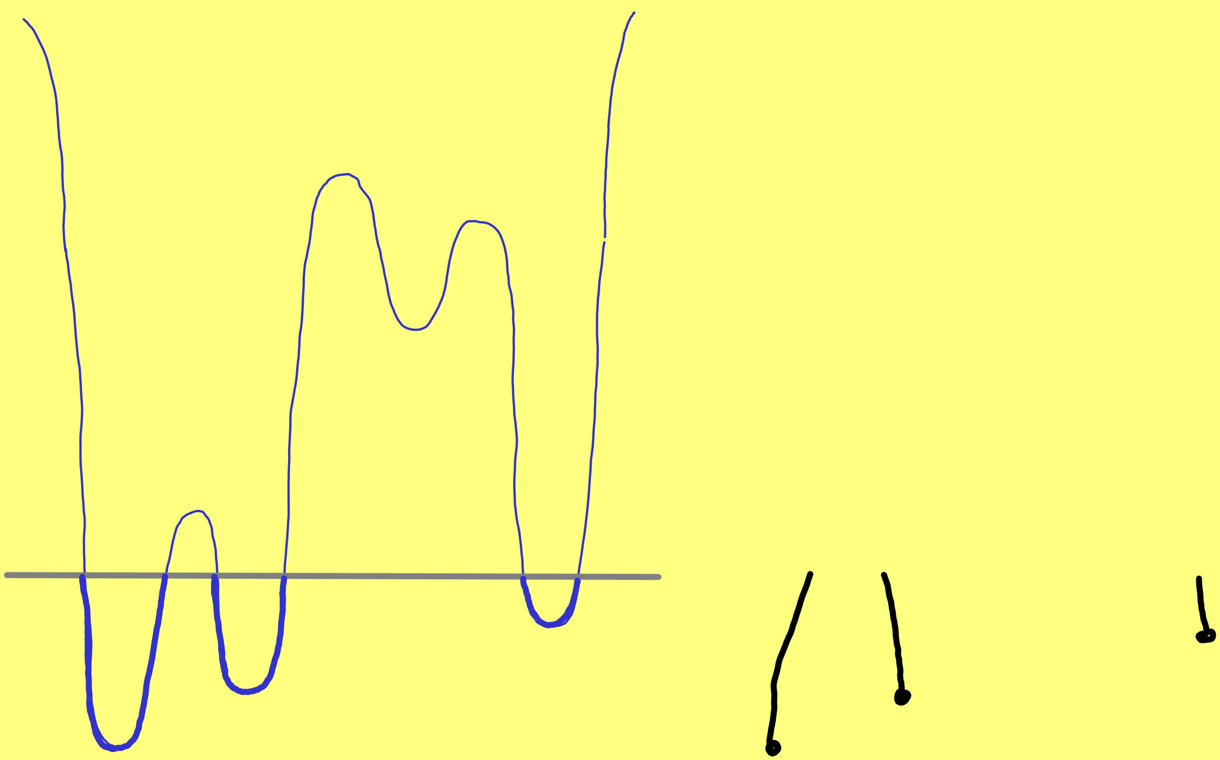
III Somites

I.1 One-D Functions

function

$f : \mathbb{S}^1 \to \mathbb{R}$

I.1 One-D Functions



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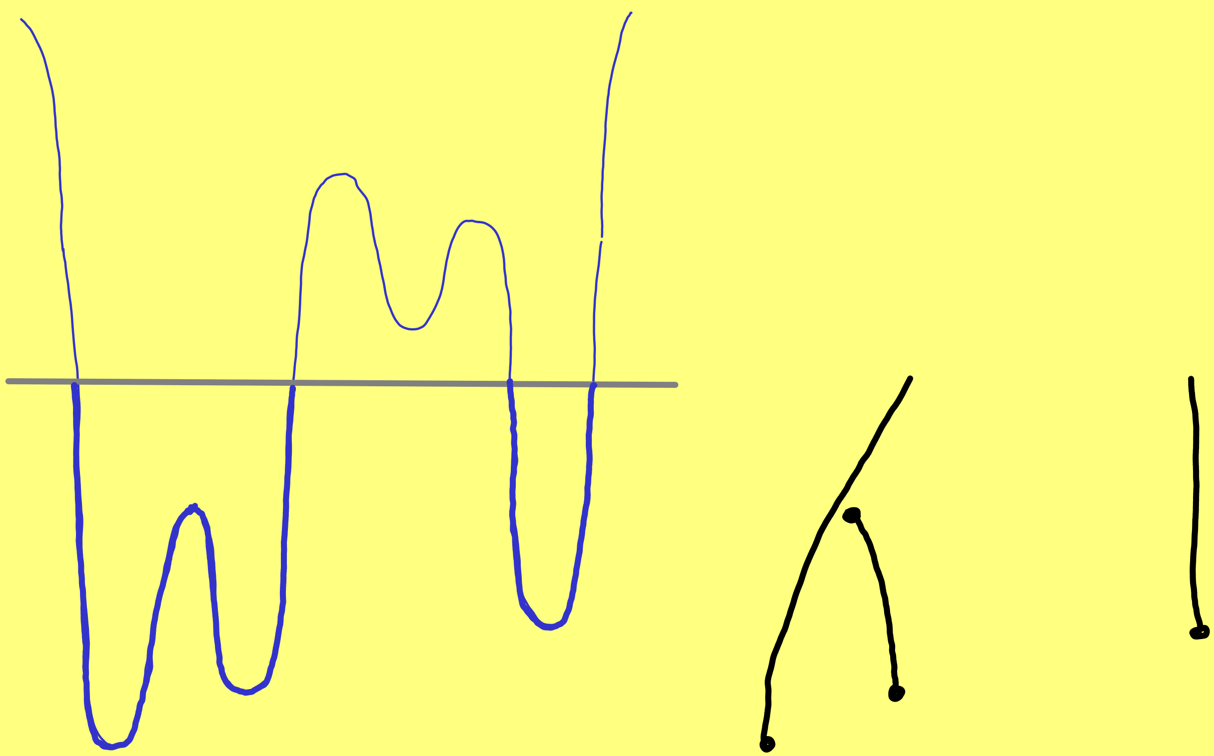

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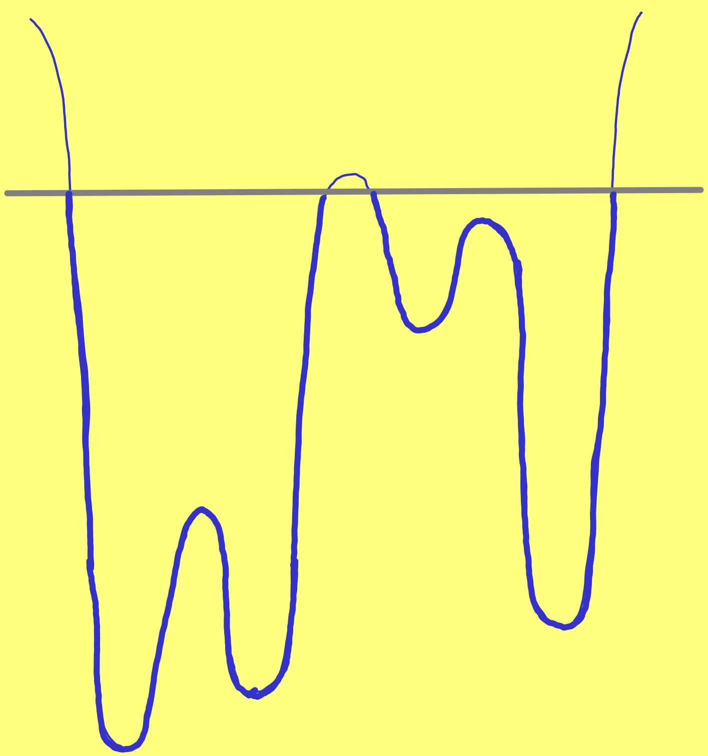

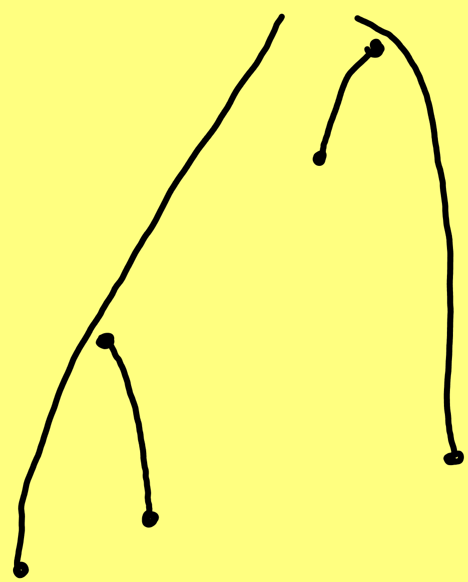

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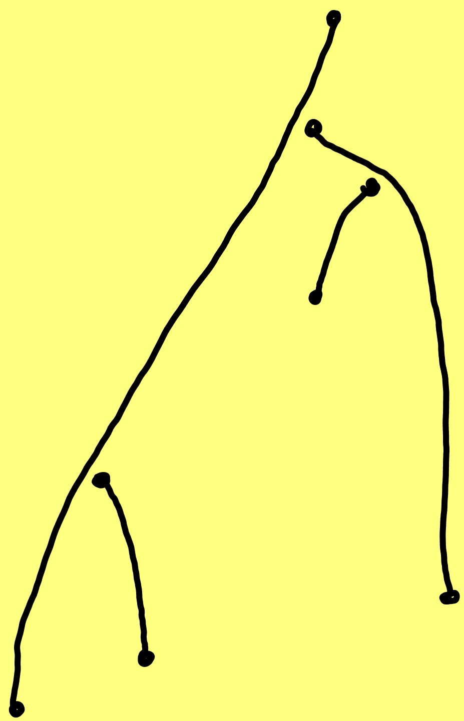

merge tree

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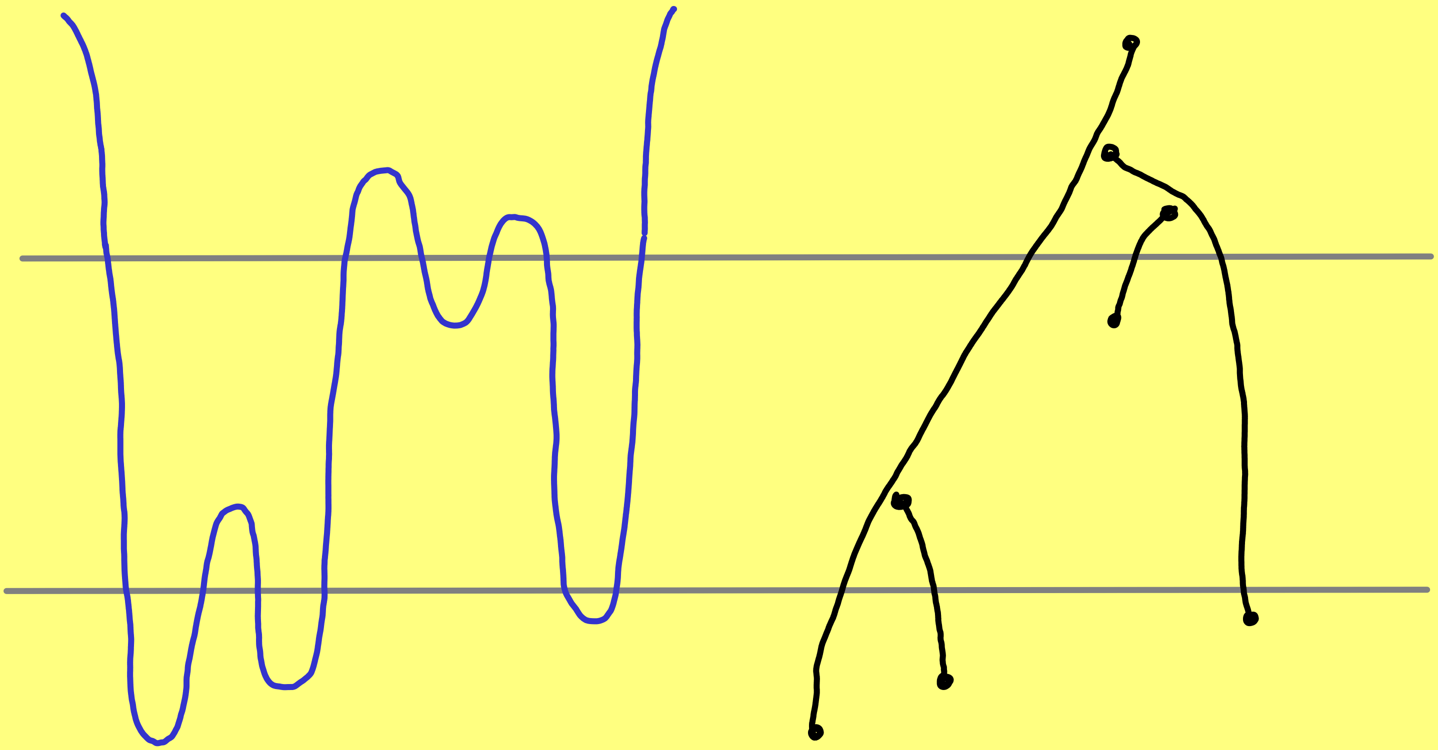

function $\quad\quad\quad\quad$ merge tree

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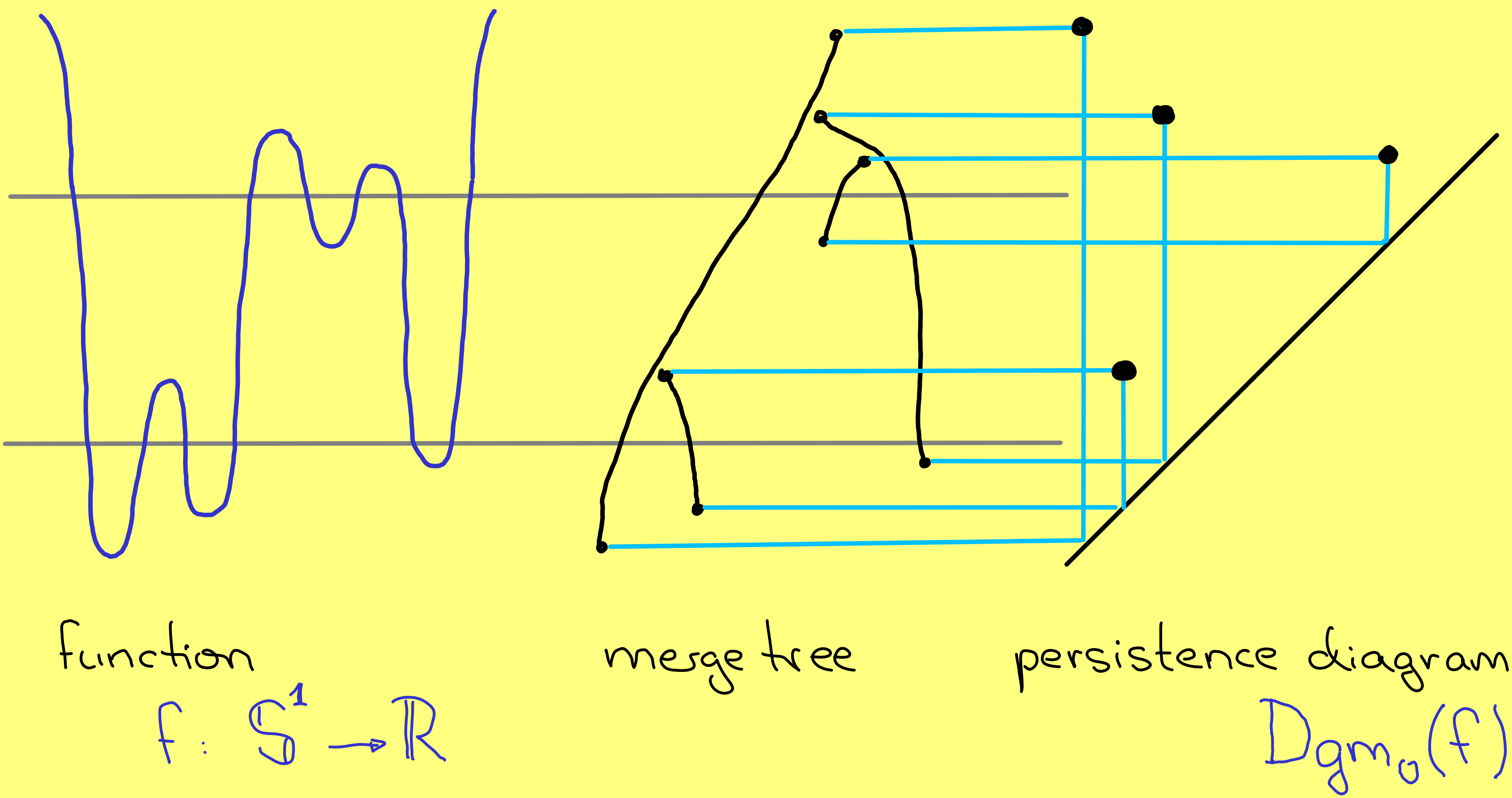

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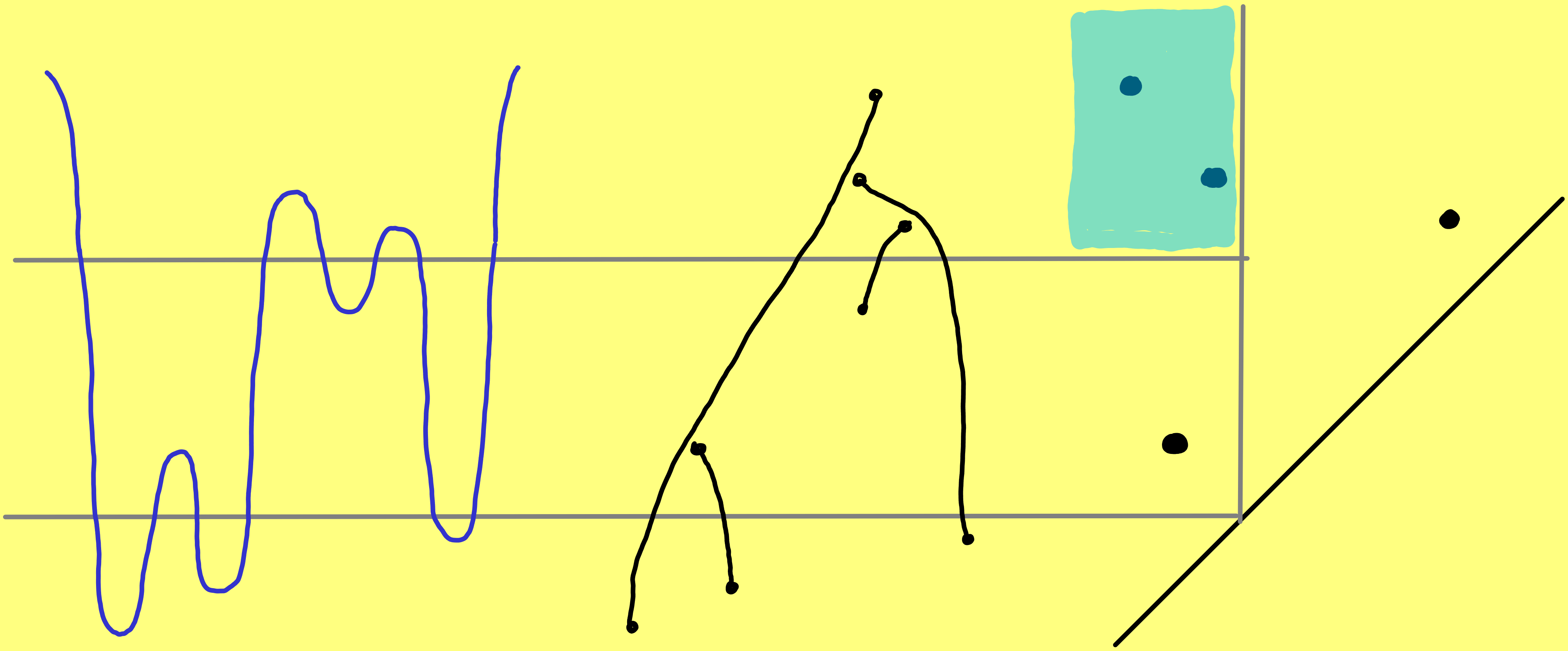


function

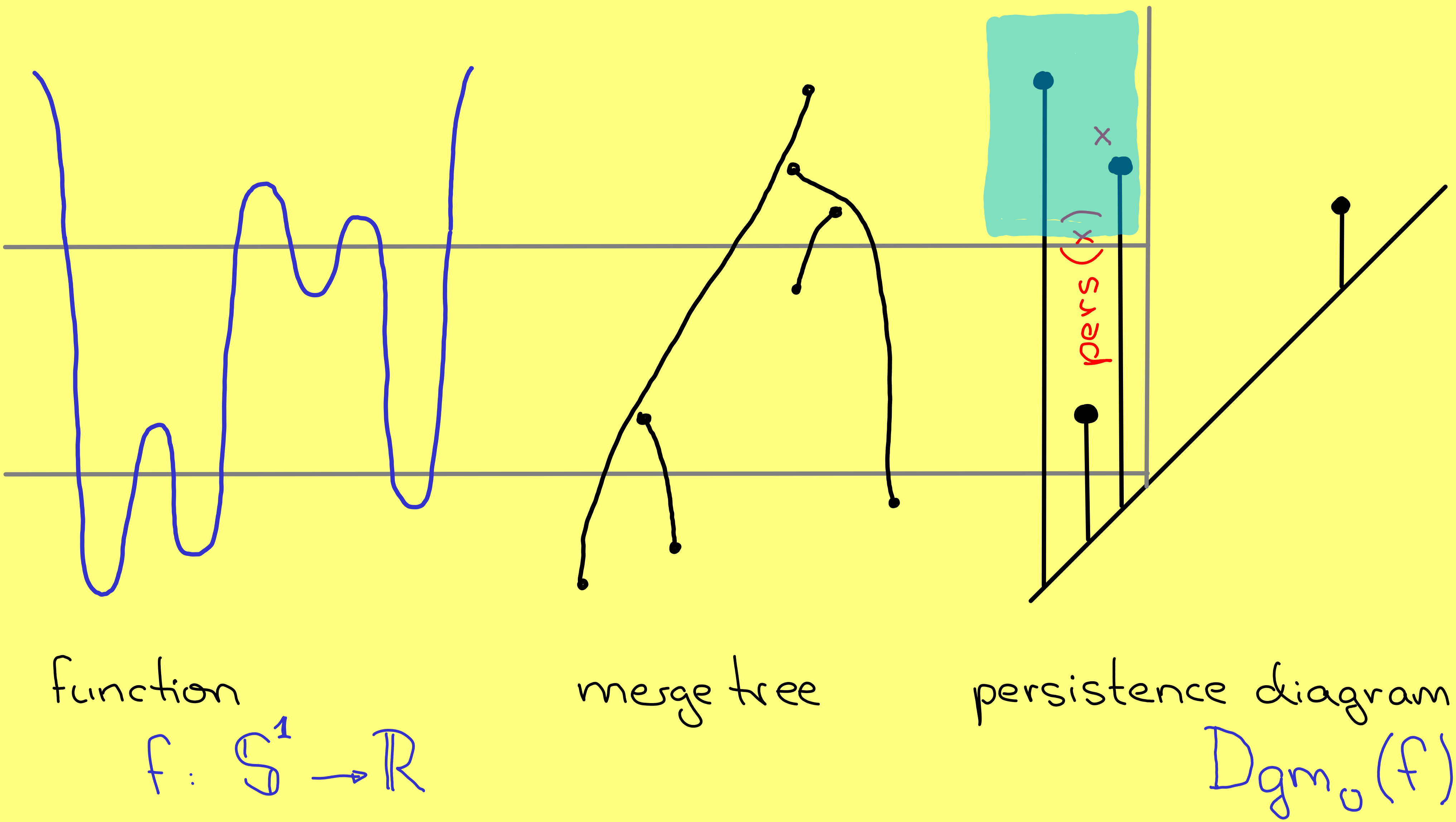
$f : \mathbb{S}^1 \to \mathbb{R}$

merge tree

persistence diagram

$\mathrm{Dgm}_0(f)$

I.1 One-D Functions



I.2 Filtrations

$$\ldots \subseteq K_{i-1} \subseteq K_i \subseteq \ldots \subseteq K_{j-1} \subseteq K_j \subseteq \ldots$$

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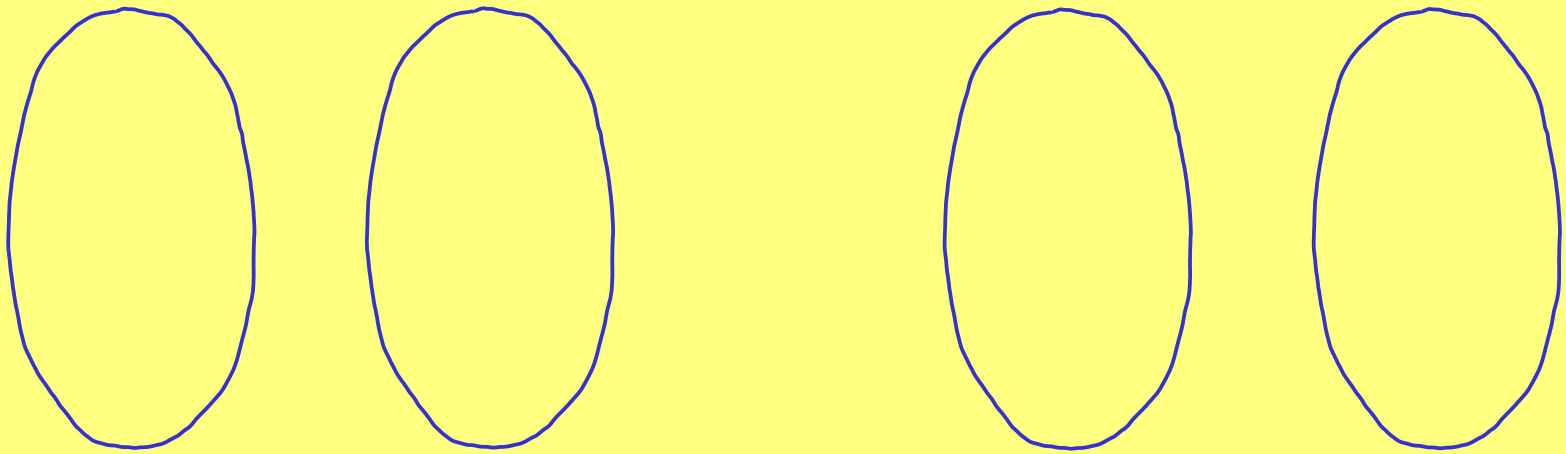

$$\cdots \longrightarrow H_p(K_{i-1}) \longrightarrow H_p(K_i) \longrightarrow \cdots \longrightarrow H_p(K_{j-1}) \longrightarrow H_p(K_j) \longrightarrow \cdots$$

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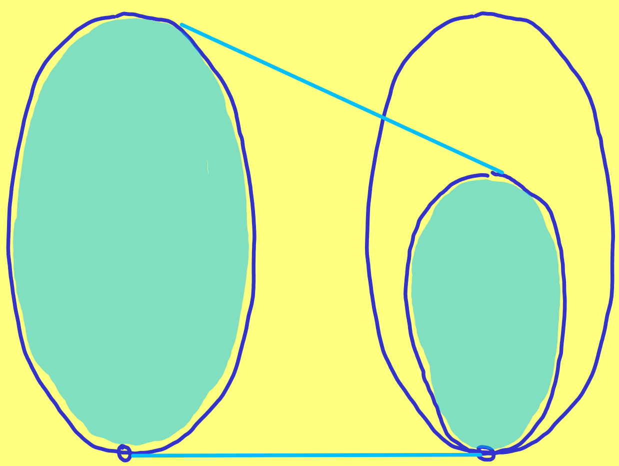

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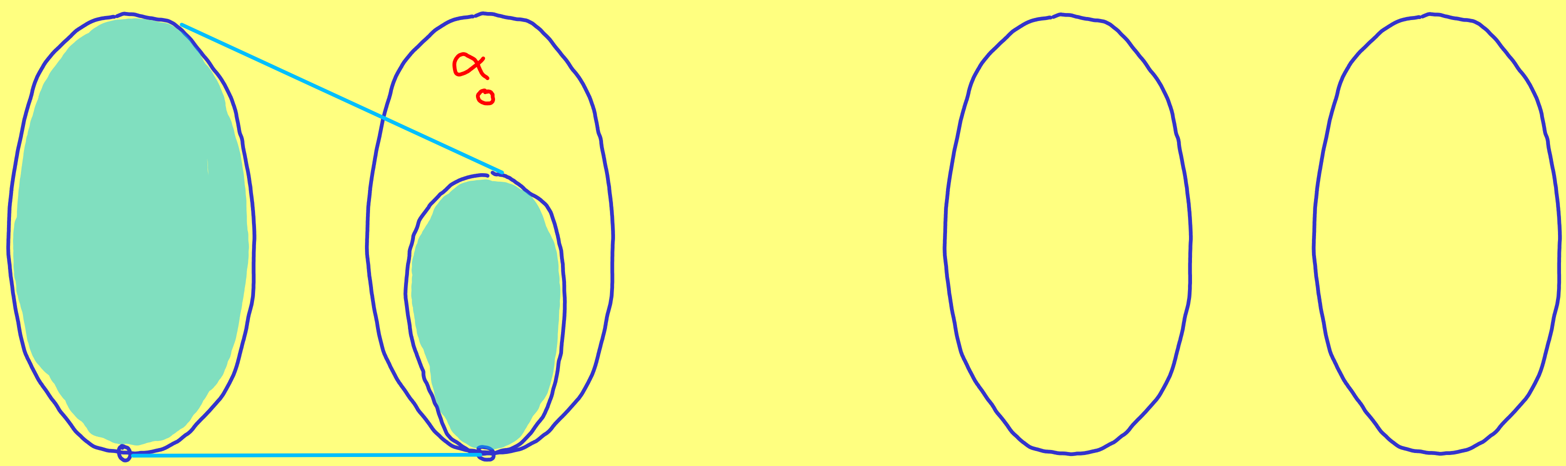

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α is born at K_i

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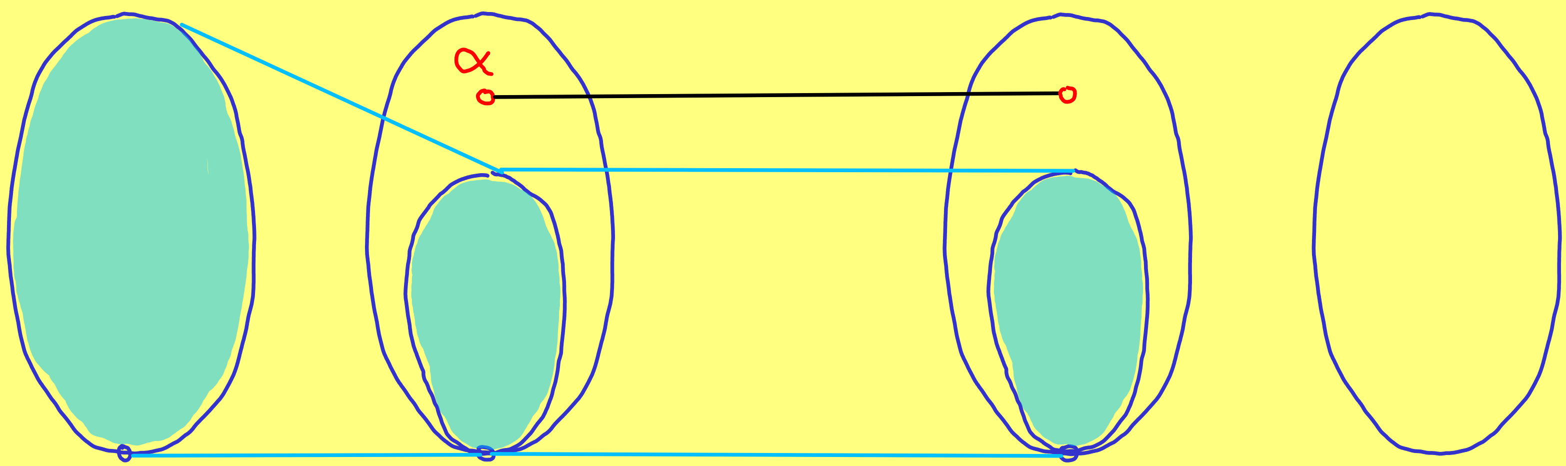

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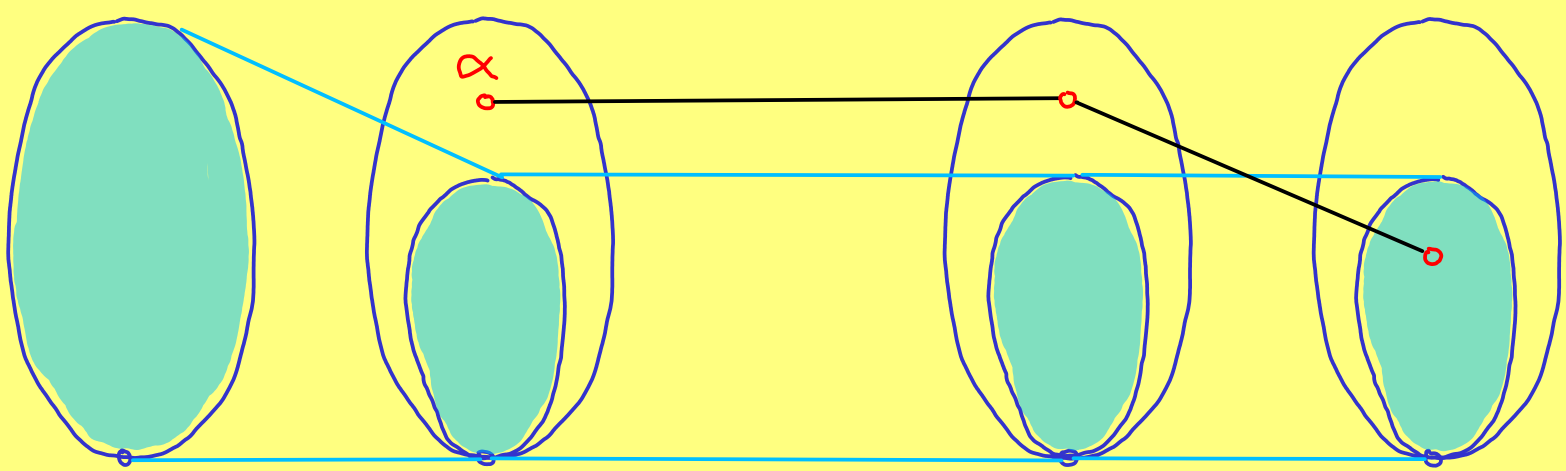

$$\ldots \to H_p(K_{i-1}) \to H_p(K_i) \to \ldots \to H_p(K_{j-1}) \to H_p(K_j) \to \ldots$$

α is born at K_i and dies entering K_j

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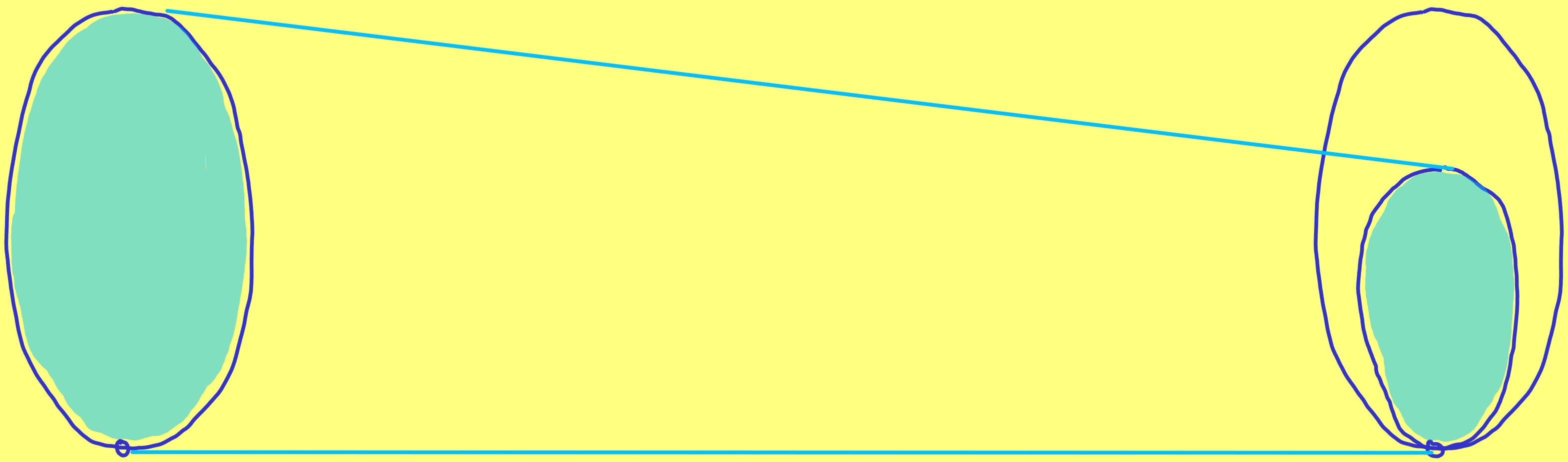
$$\dots \subseteq K_{i-1} \subseteq K_i \subseteq \dots \subseteq K_{j-1} \subseteq K_j \subseteq \dots$$



$$H_p(K_{i-1}) \to \quad \dots \quad \to H_p(K_j)$$

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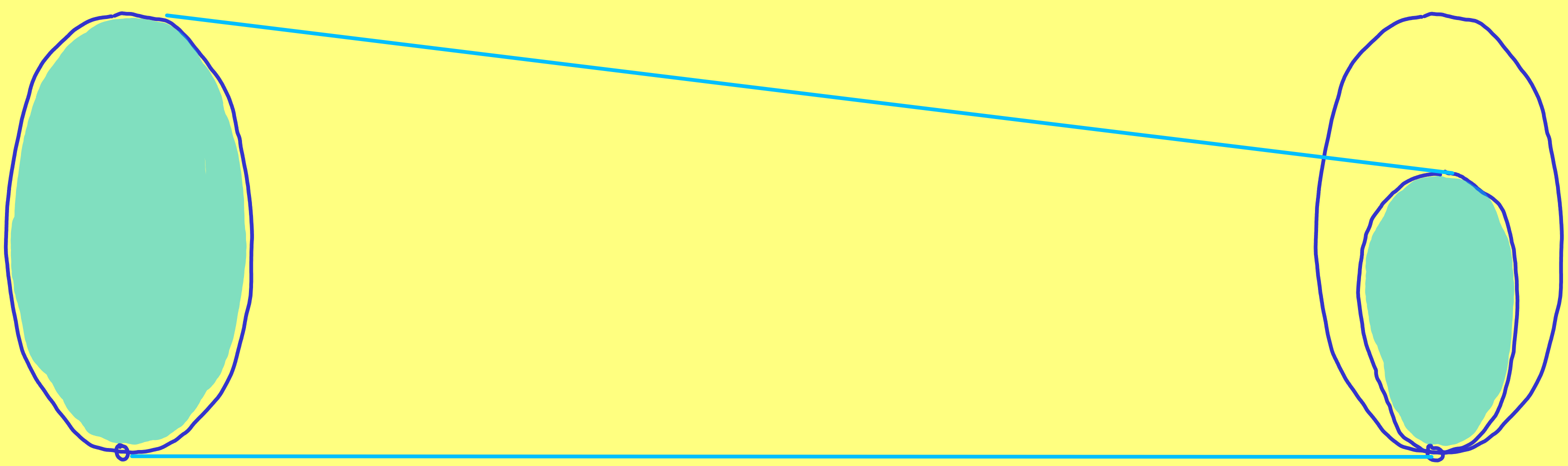
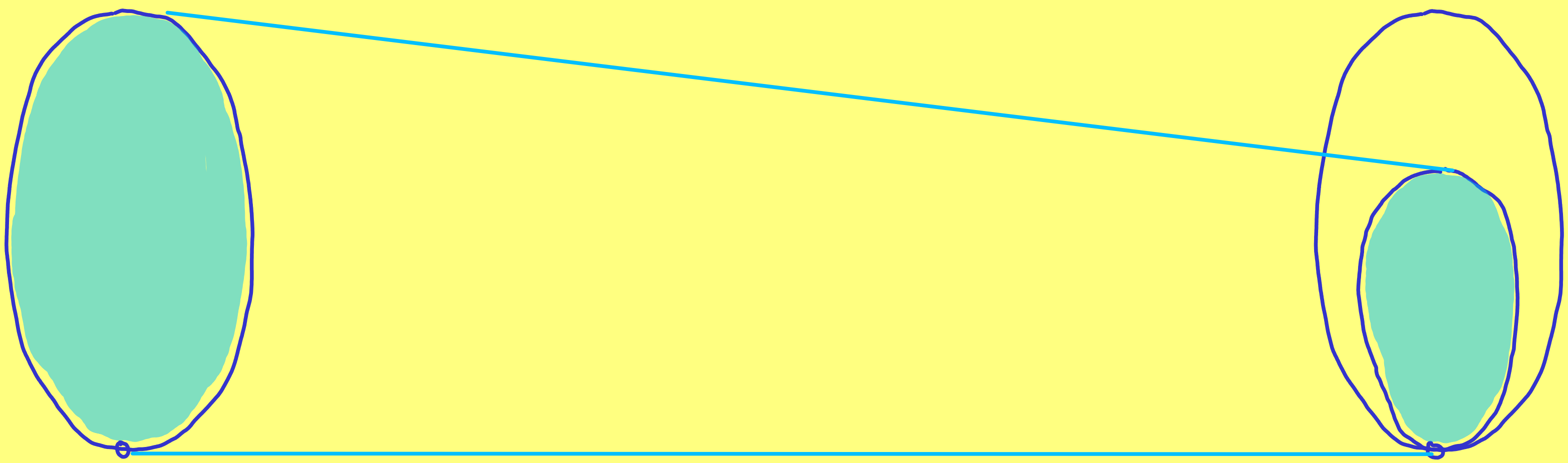


$$h_p^{i-1,j} : H_p(K_{i-1}) \longrightarrow \quad \ldots \quad \longrightarrow H_p(K_j)$$

induced homomorphism

I.2 Filtrations

$$\ldots \subseteq K_{i-1} \subseteq K_i \subseteq \ldots \subseteq K_{j-1} \subseteq K_j \subseteq \ldots$$



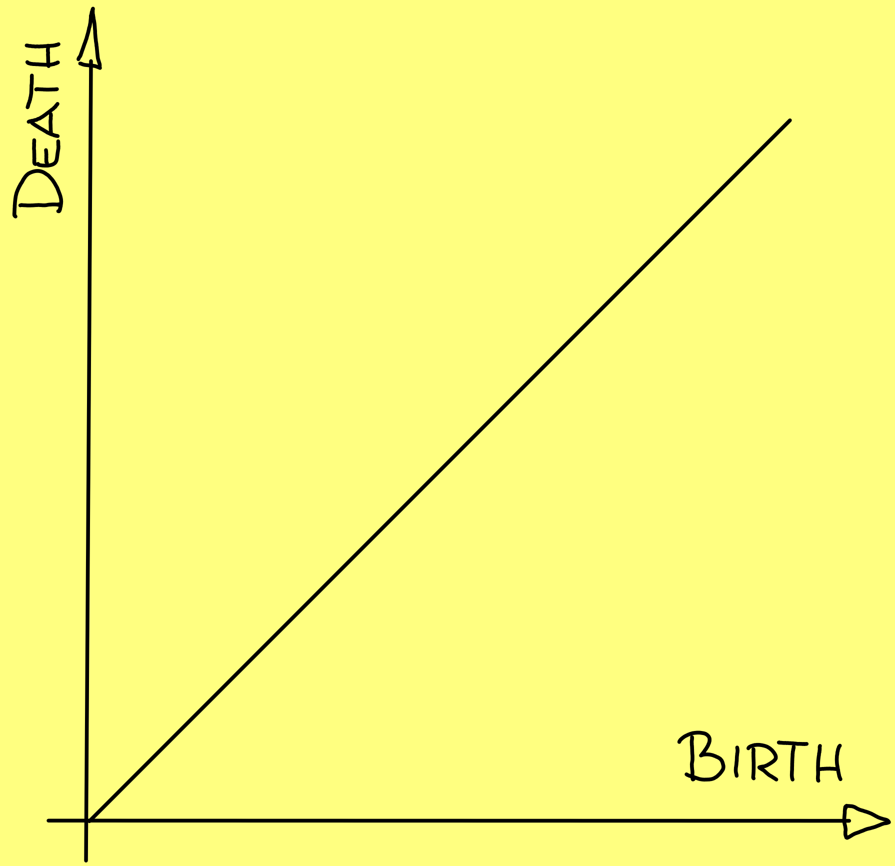
$$h_p^{i-1,j} : H_p(K_{i-1}) \to \ldots \to H_p(K_j)$$

induced homomorphism

$\mathrm{im}\, h_p^{i-1,j}$ is persistent homology group

I.3 Diagrams

$\mathrm{Dgm}_p(f)$:



I.3 Diagrams

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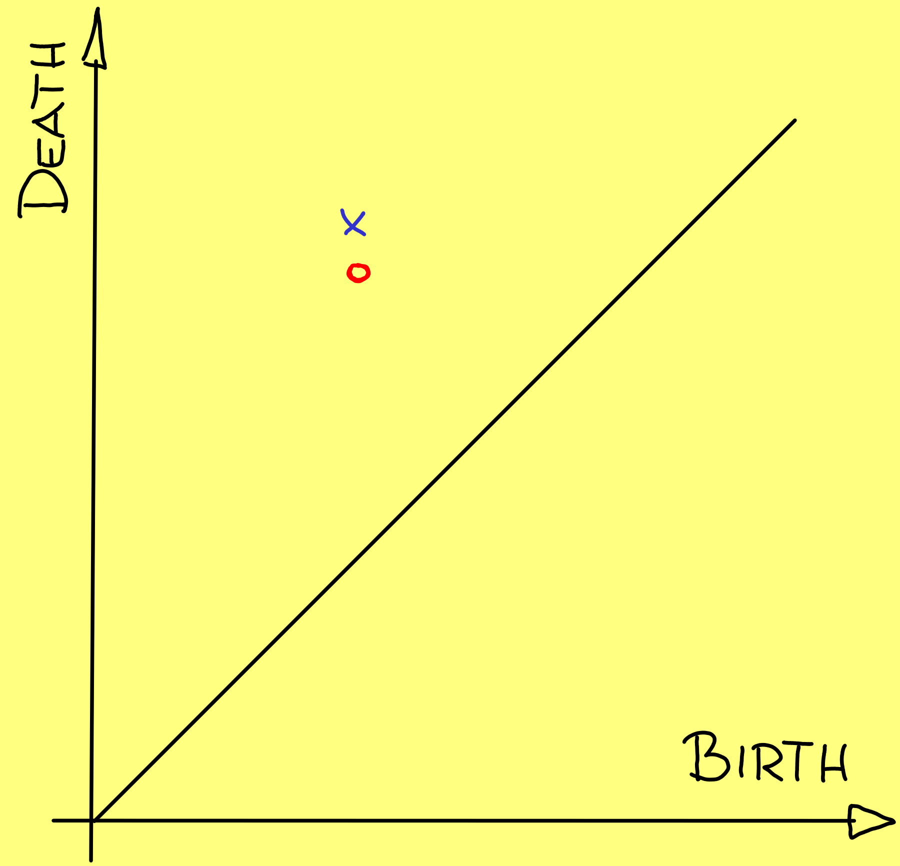

I.3 DIAGRAMS

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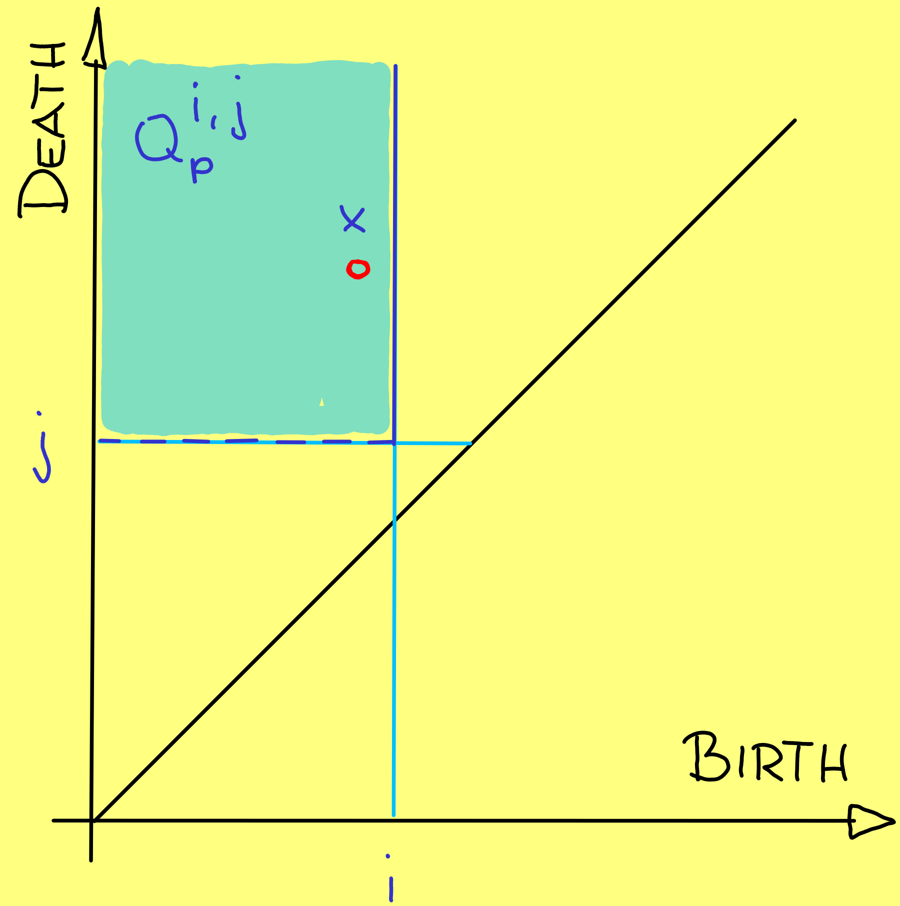

I.3 DIAGRAMS

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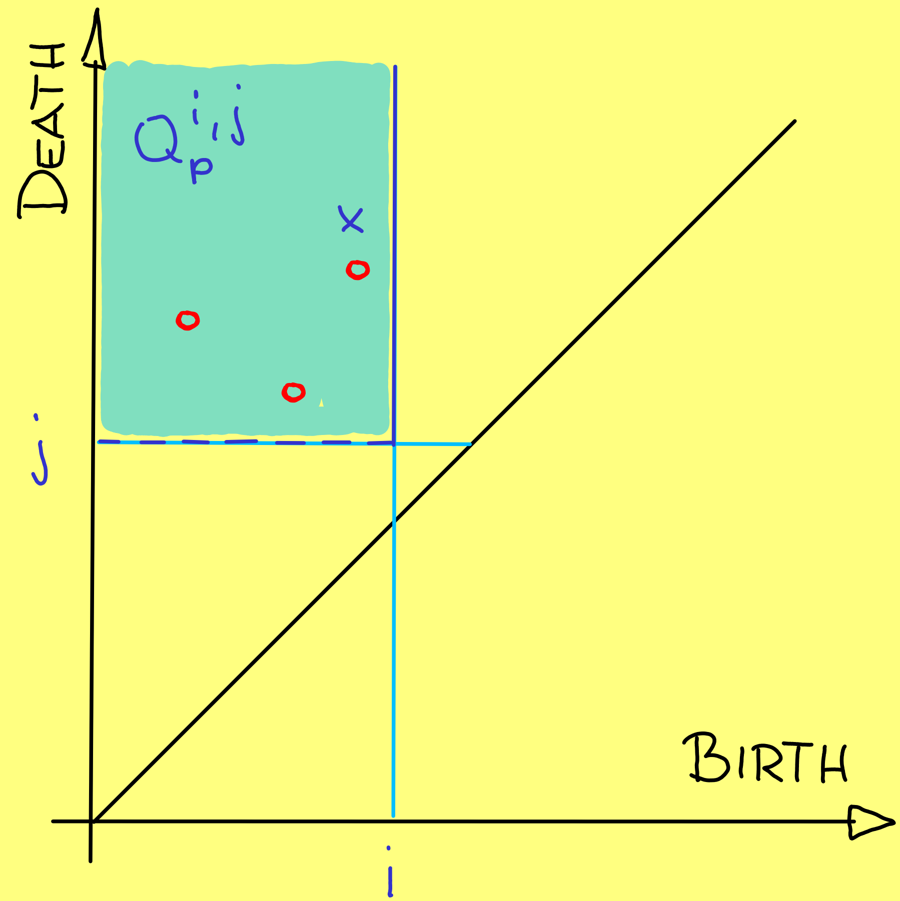


FUNDAMENTAL LEMMA OF PERSISTENT HOMOLOGY.

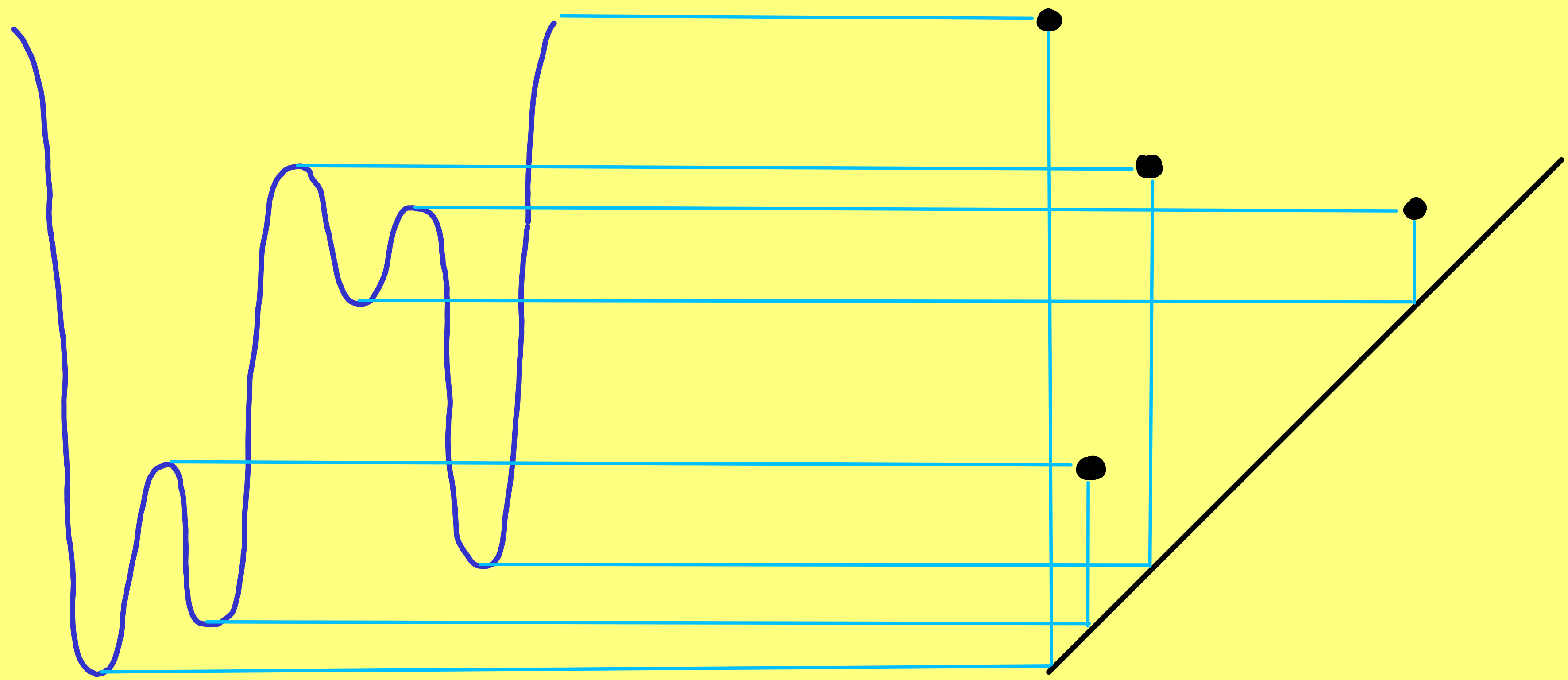
#points in $Q_p^{i,j}$ = rank im $h_p^{i,j}$.

I Persistence

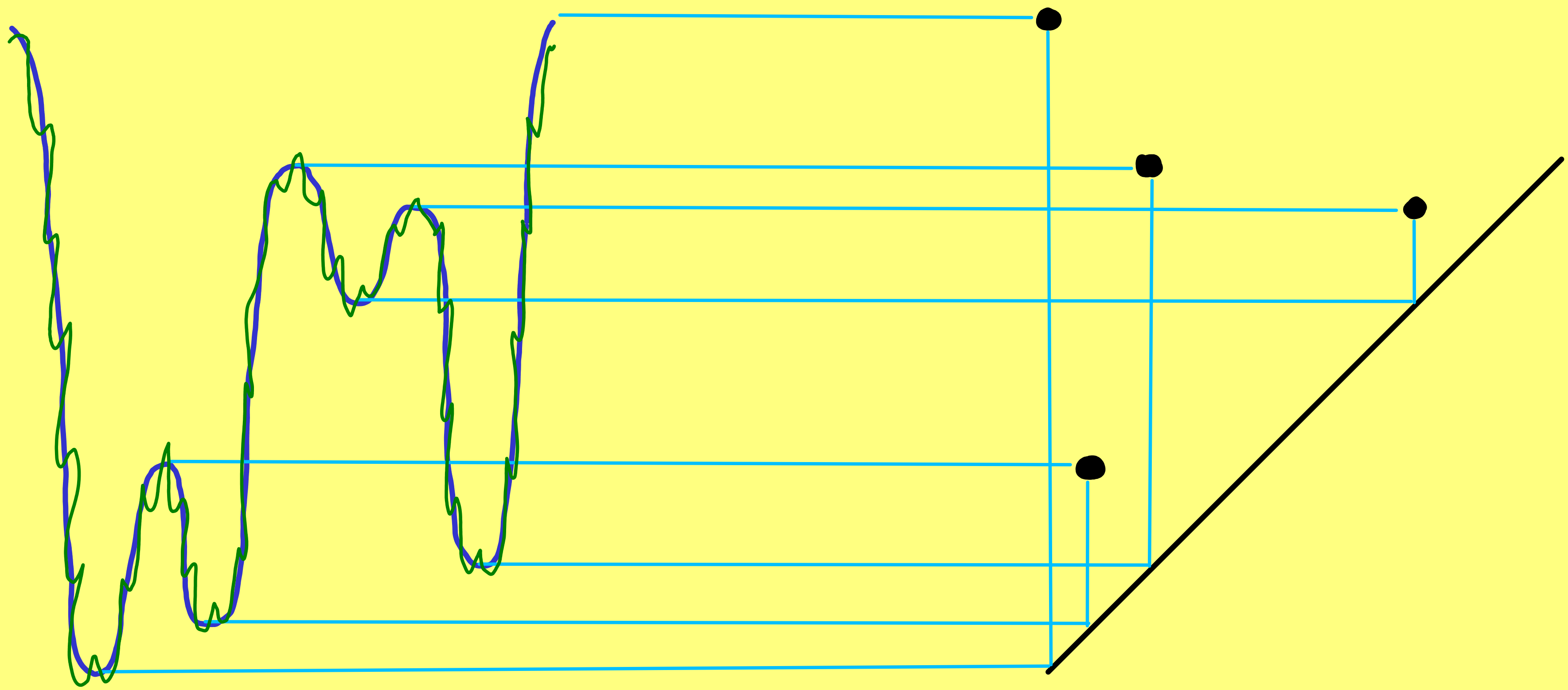
II Curves

III Somites

II.1 Bottleneck Distance



II.1 Bottleneck Distance



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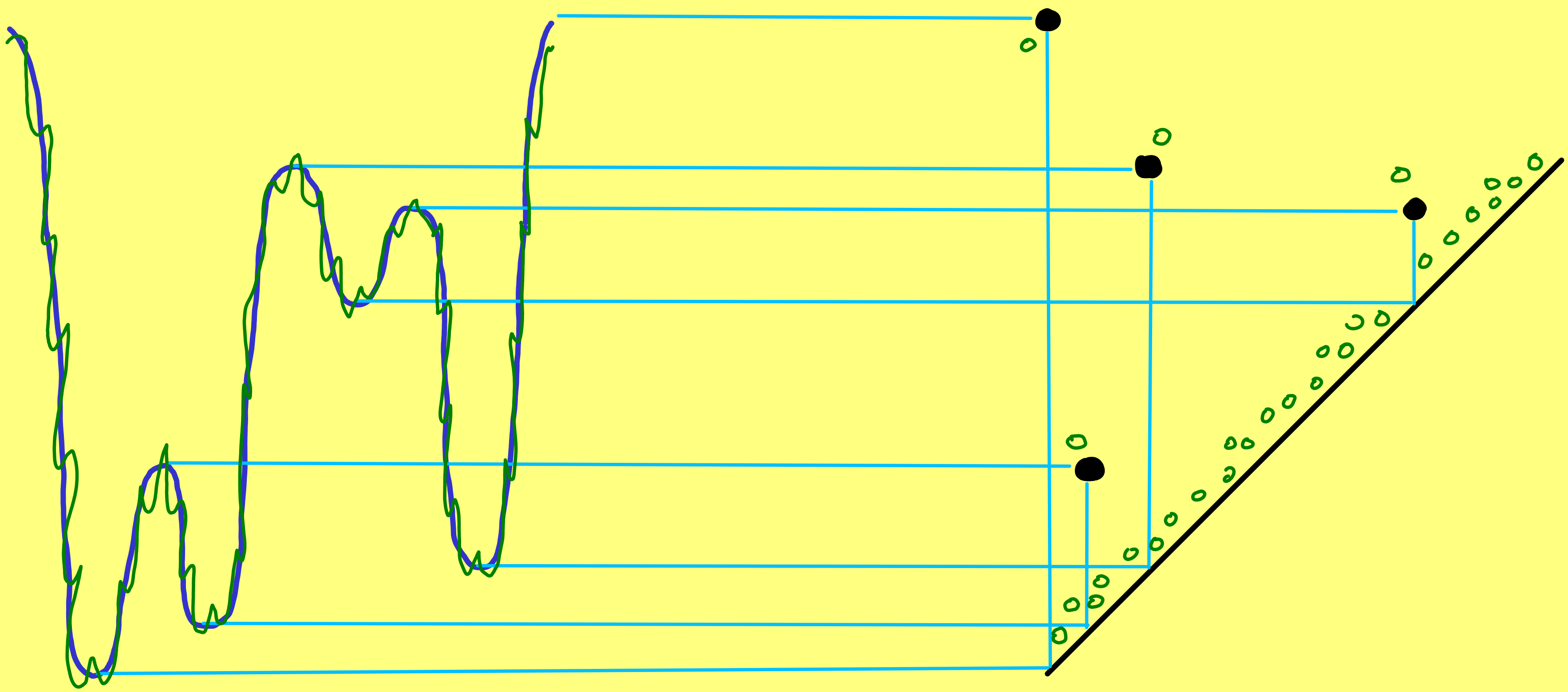

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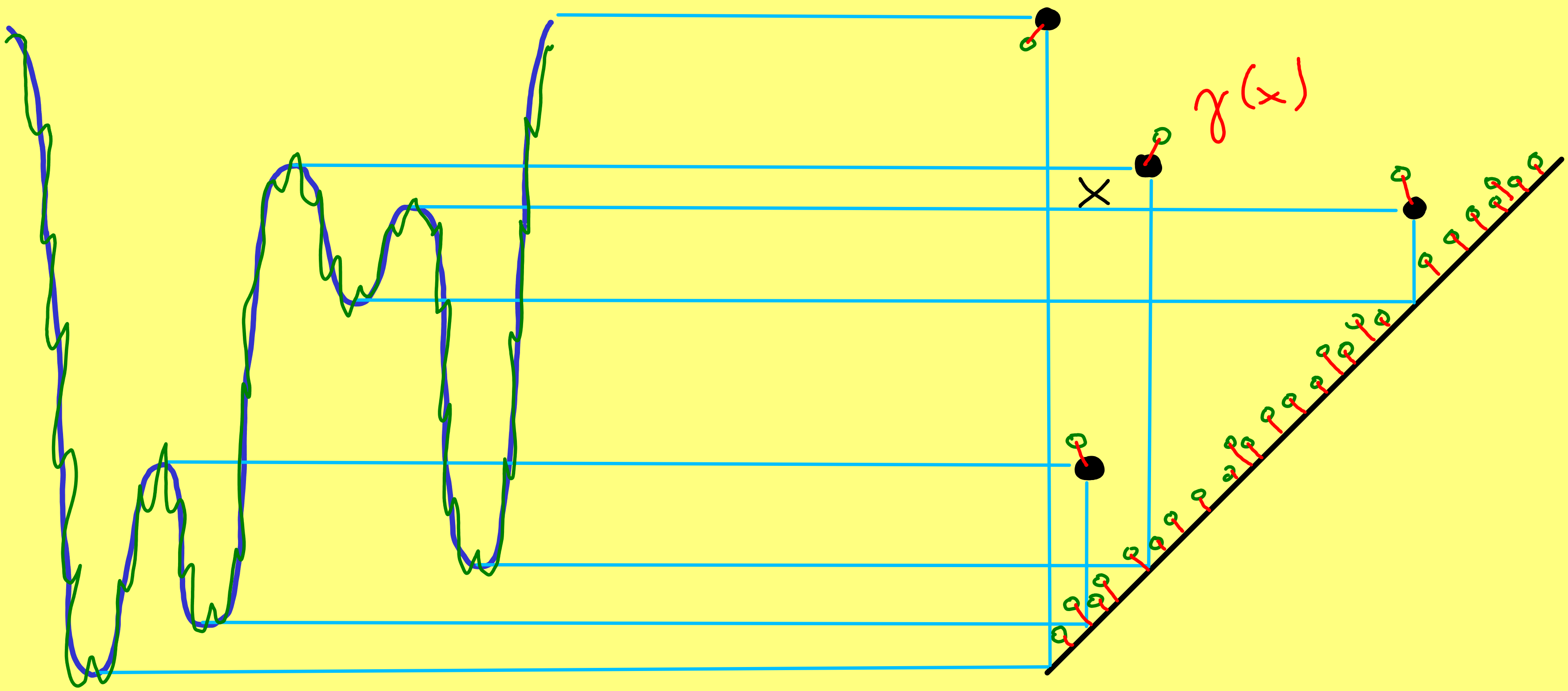

The bottleneck distance between the diagrams is

$$W_\infty\left(\mathrm{Dgm}_p(f), \mathrm{Dgm}_p(g)\right) = \inf_{\gamma} \sup_{x} \|x - \gamma(x)\|_\infty .$$

II.1 BOTTLENECK DISTANCE

L_∞-STAB. THM. For tame functions $f, g : \mathbb{X} \to \mathbb{R}$ the bottleneck distance between their diagrams is bounded by the max-difference between the functions,

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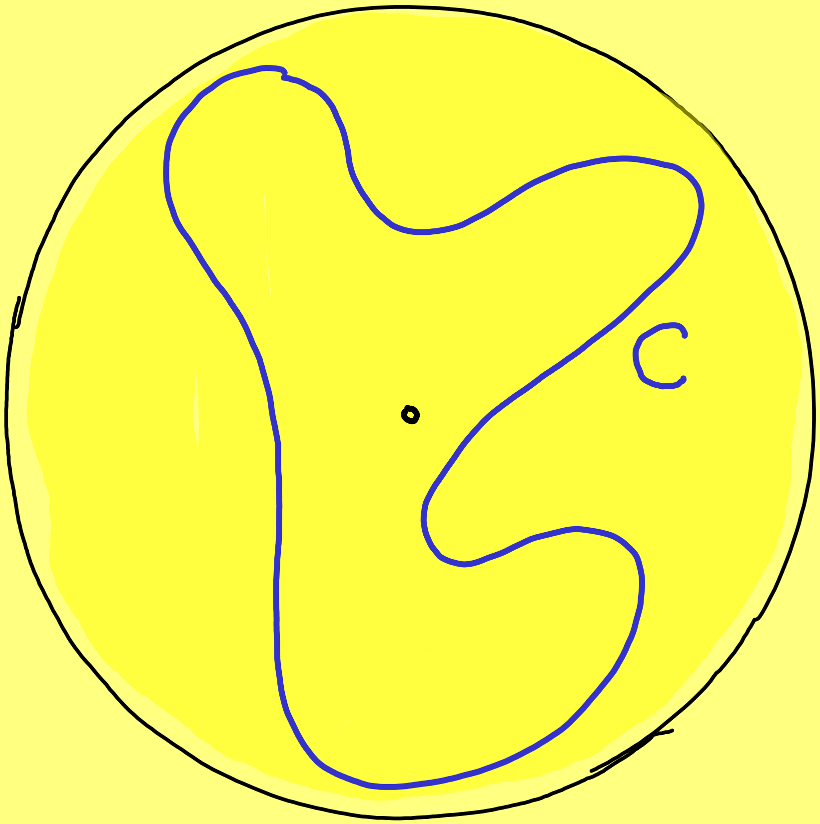
II.2 FÁRY THEOREM



C = smooth curve inside unit disk

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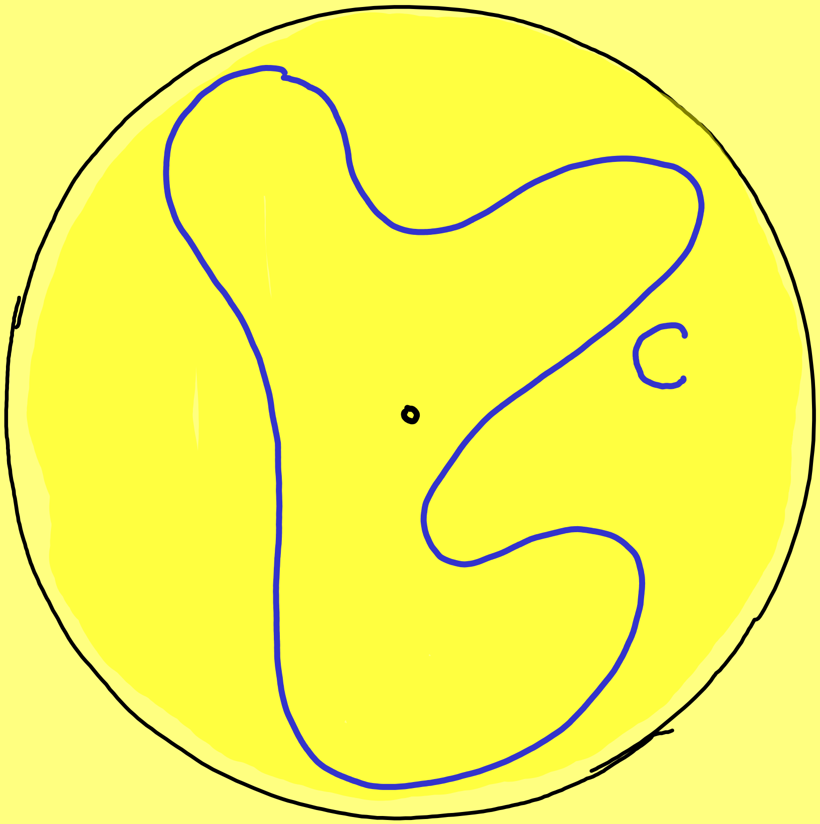

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$L(C)$ = length

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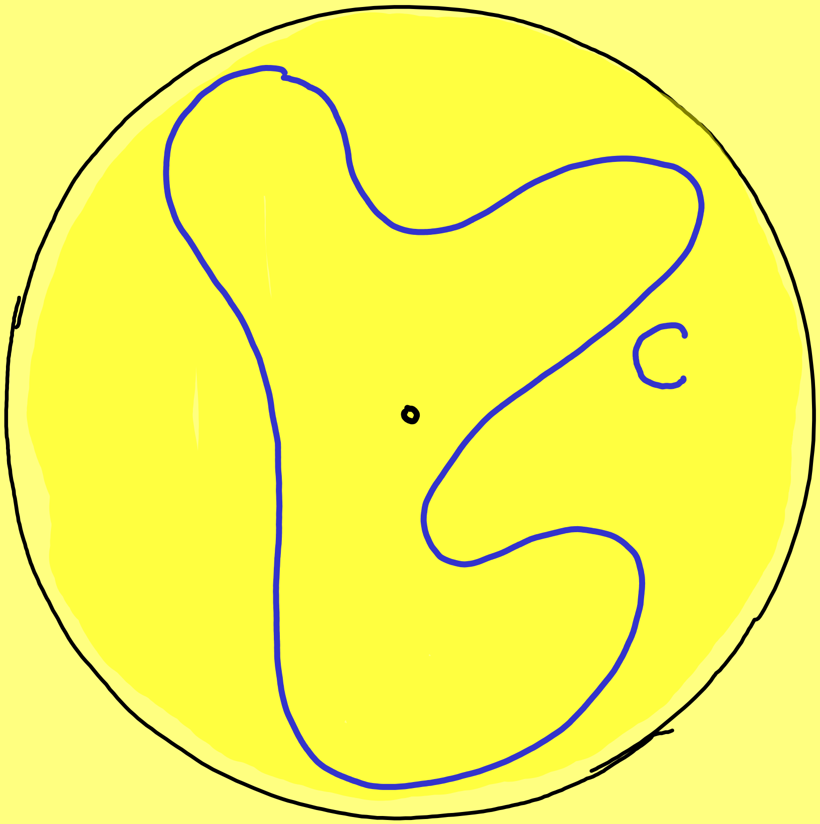

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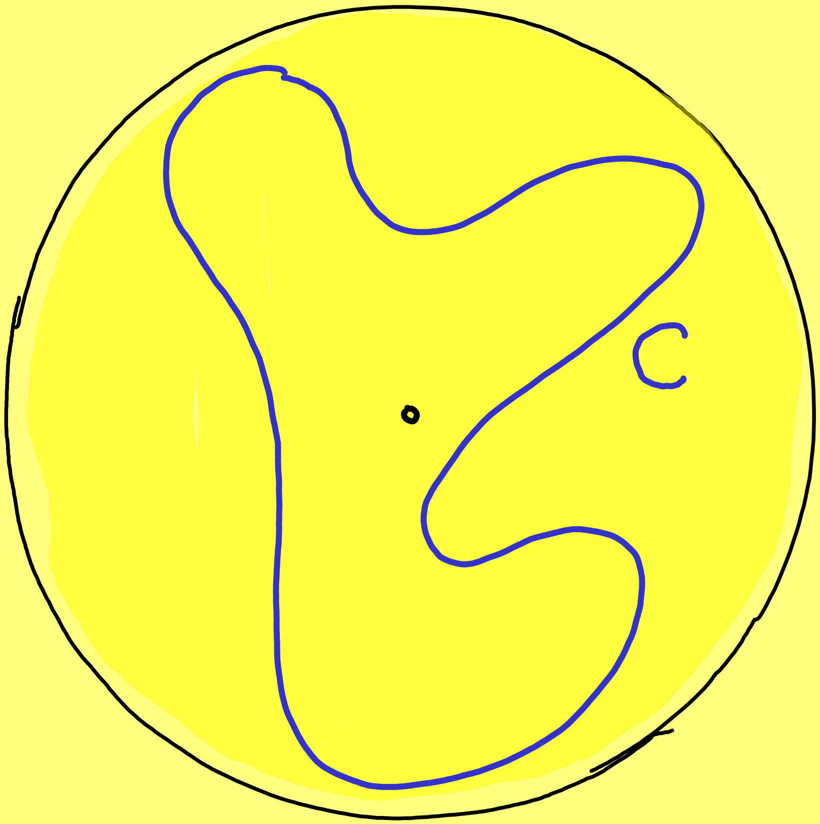

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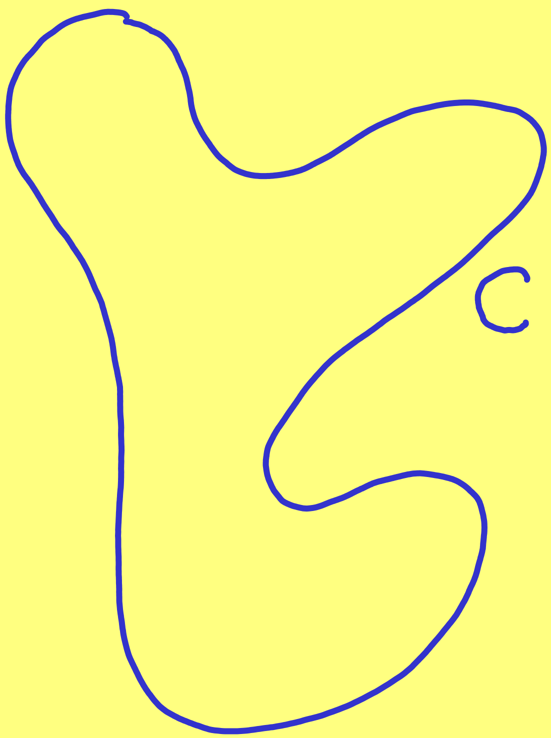
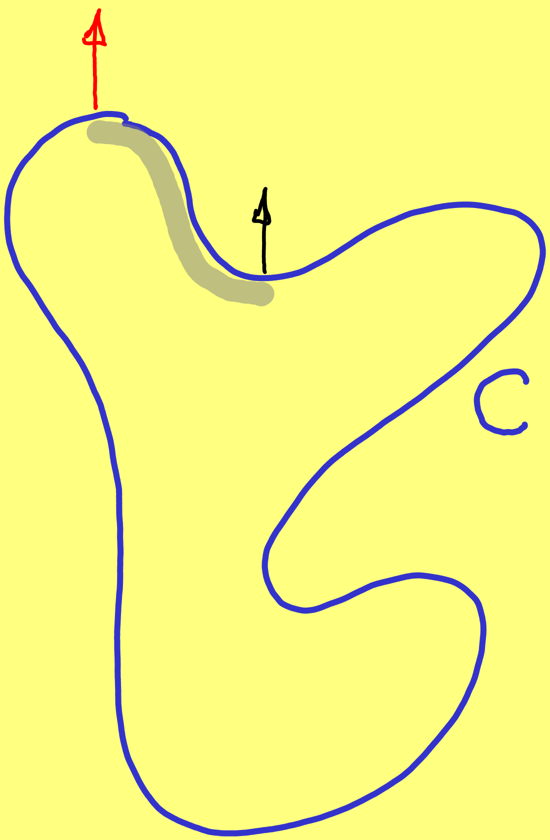
FÁRY: $L(C) \leq K(C)$.

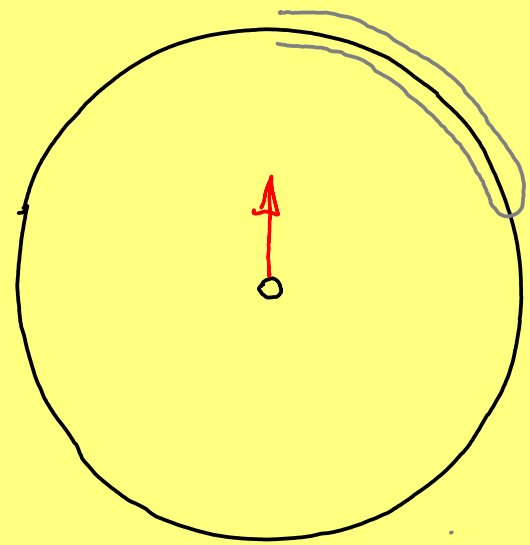
II.3 Total Curvature



$$K(C) = \int_{x \in C} |\kappa(x)|\, dx$$

II.3 Total Curvature





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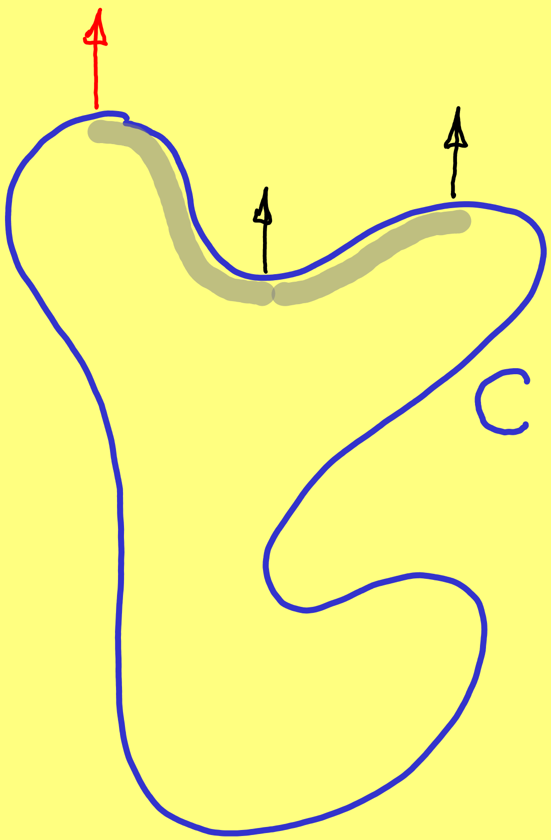

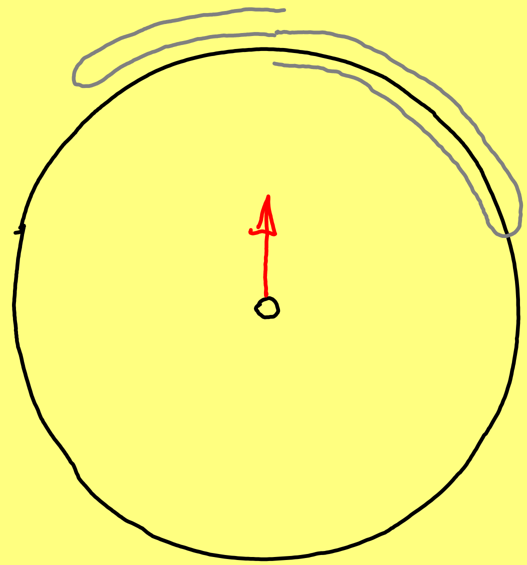

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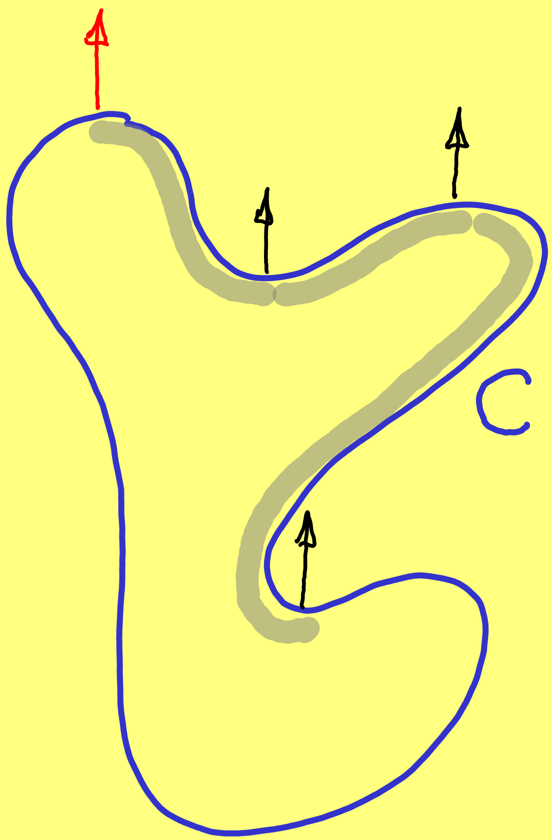
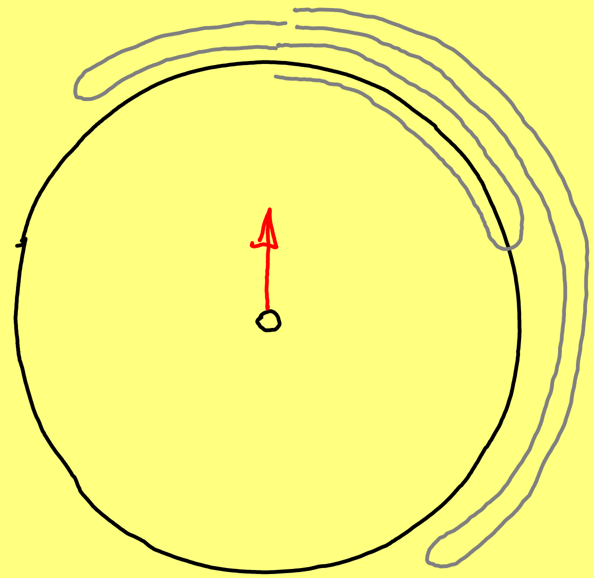
II.3 TOTAL CURVATURE





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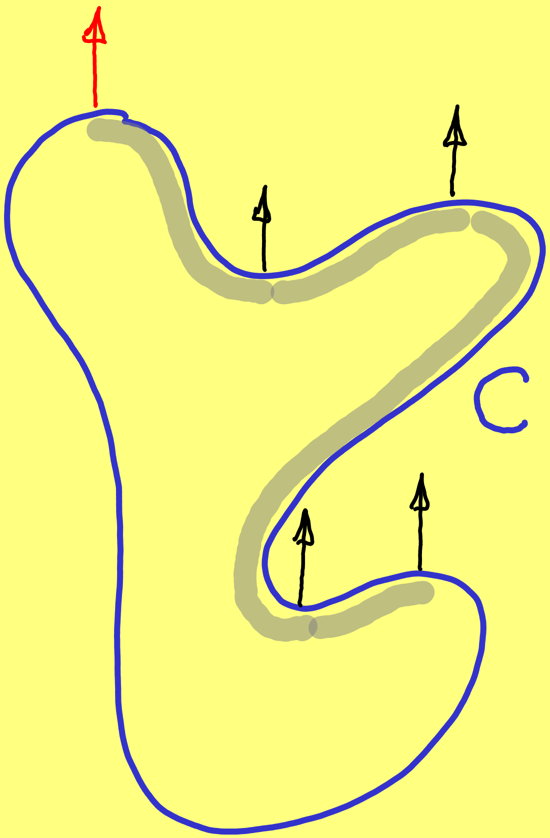

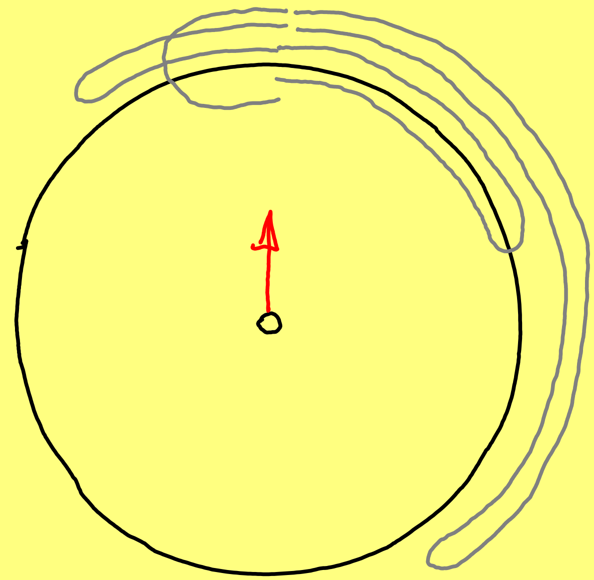

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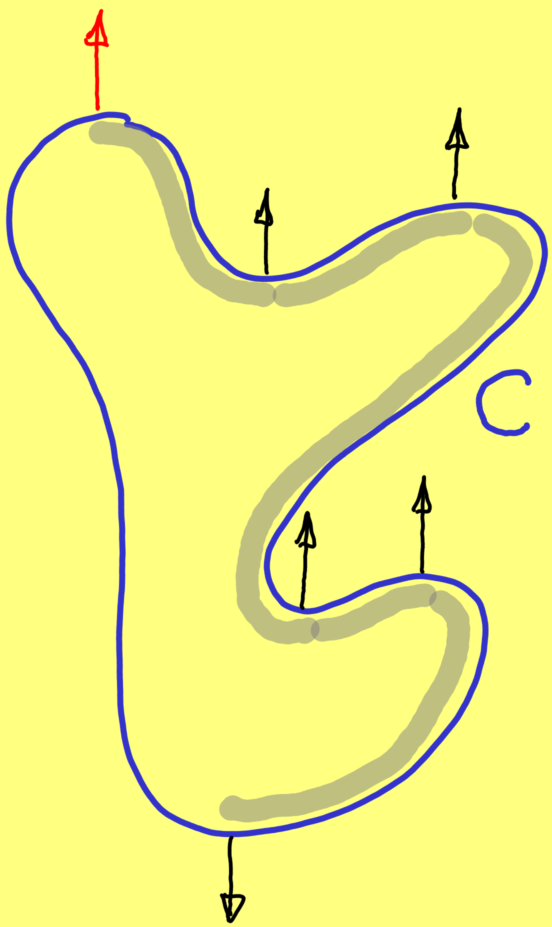

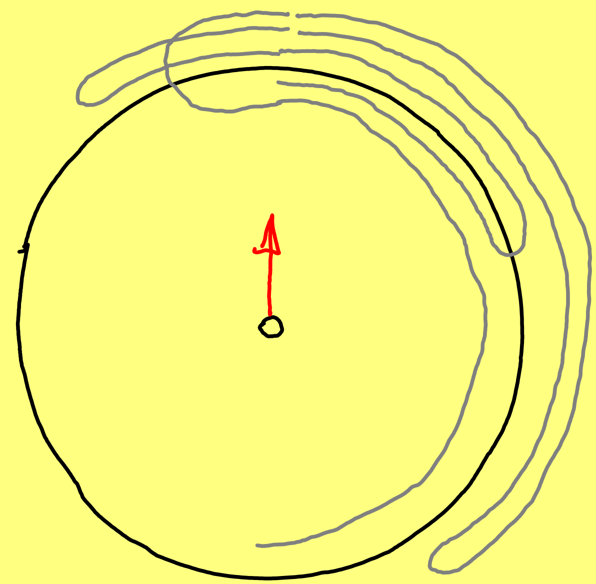

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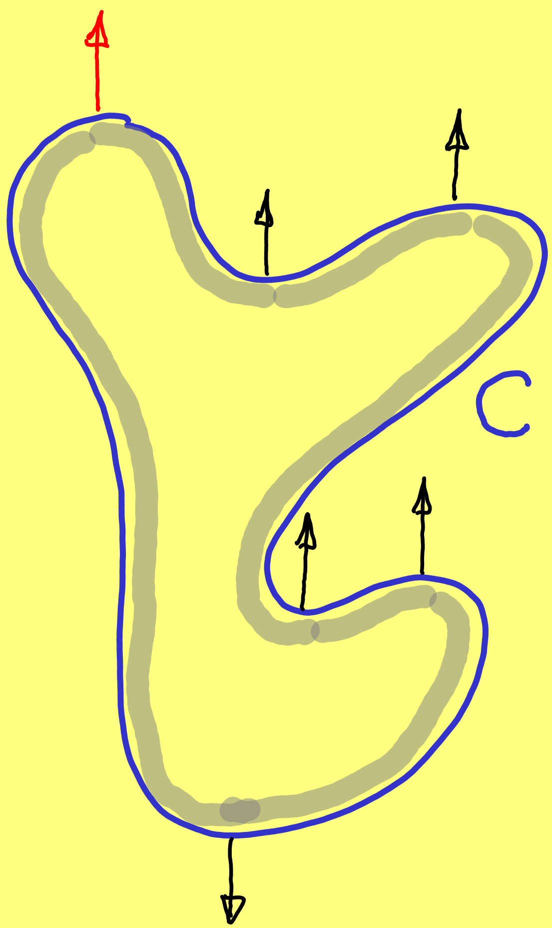

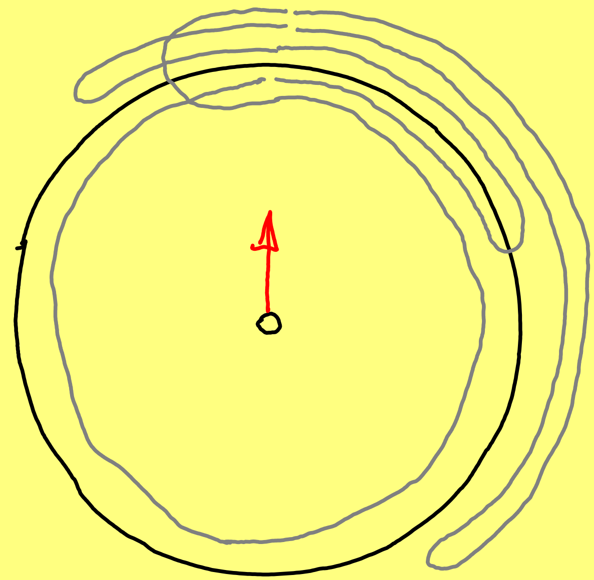

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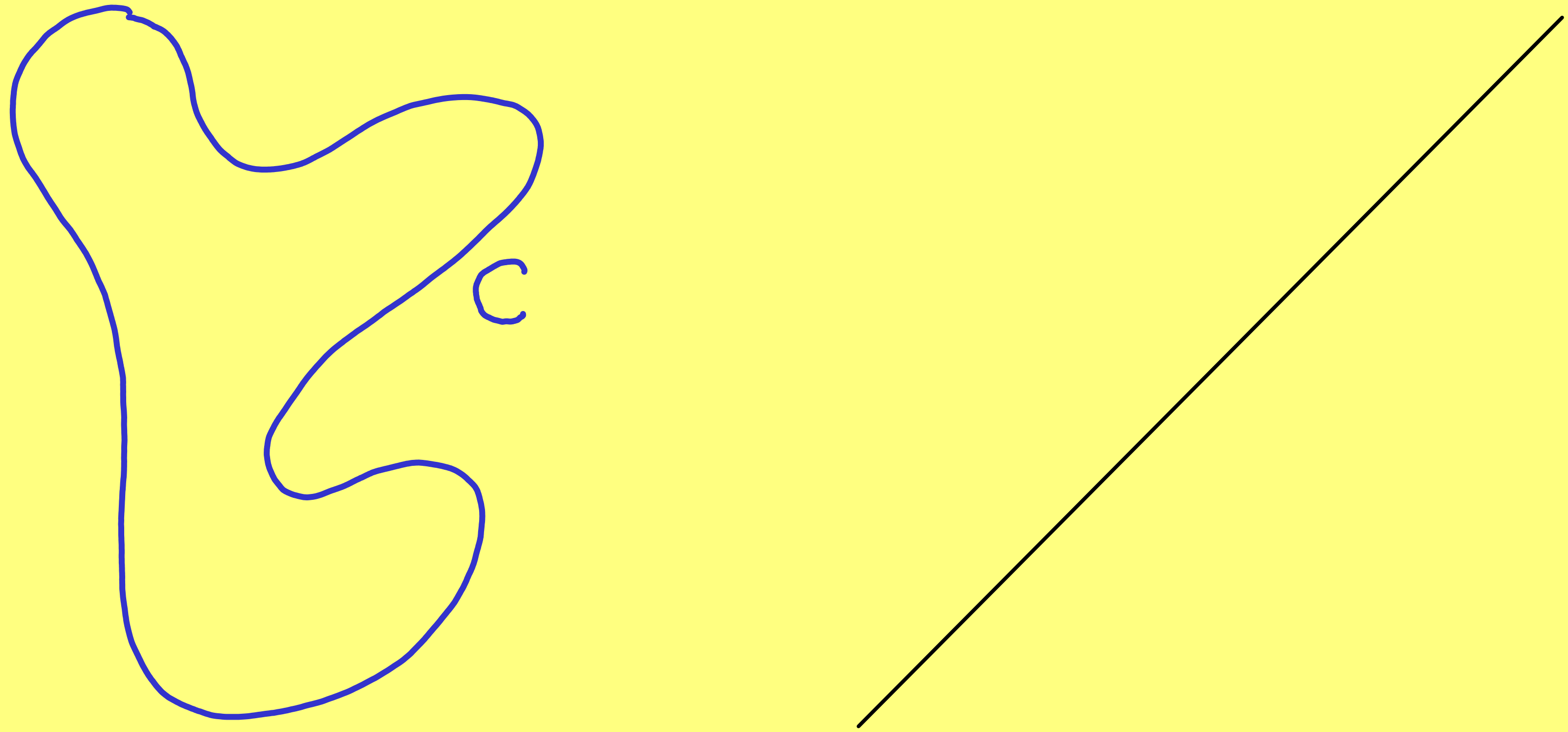
II.4 Generalized Fáry Theorem



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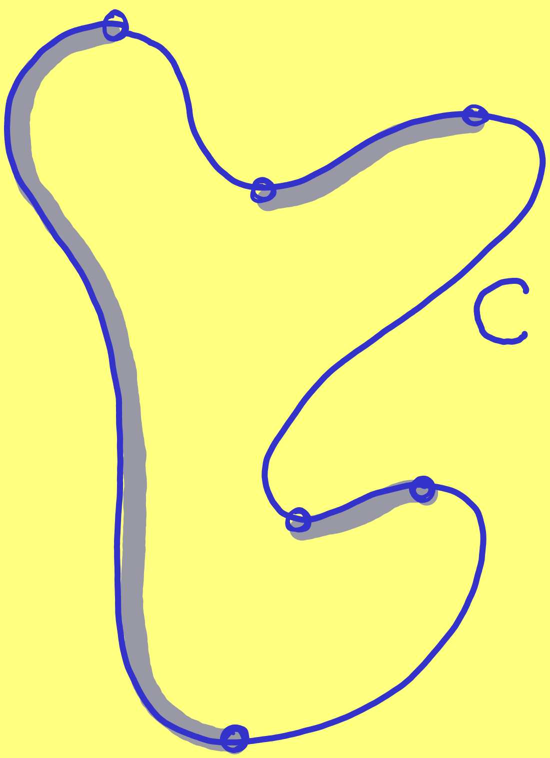

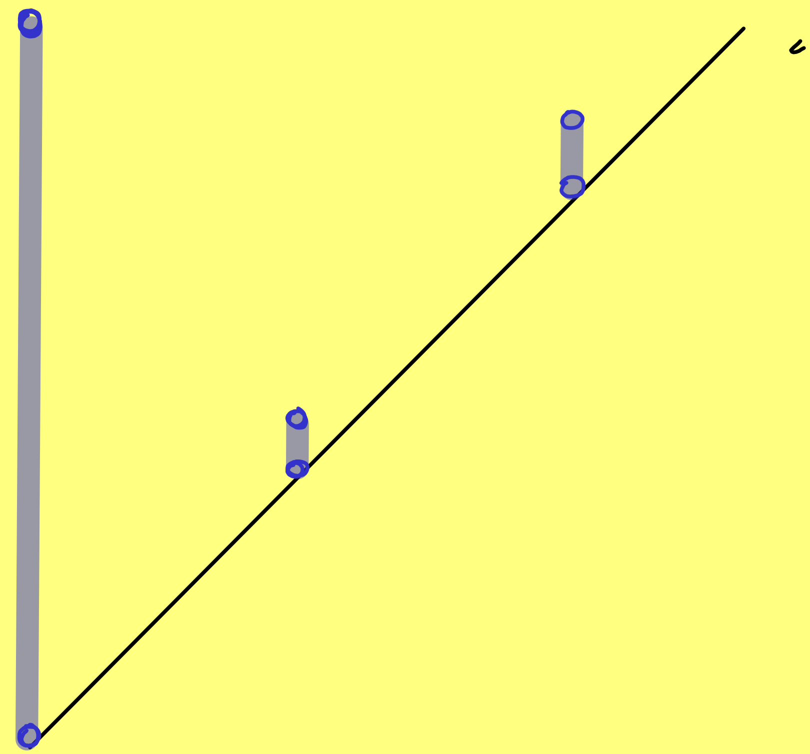

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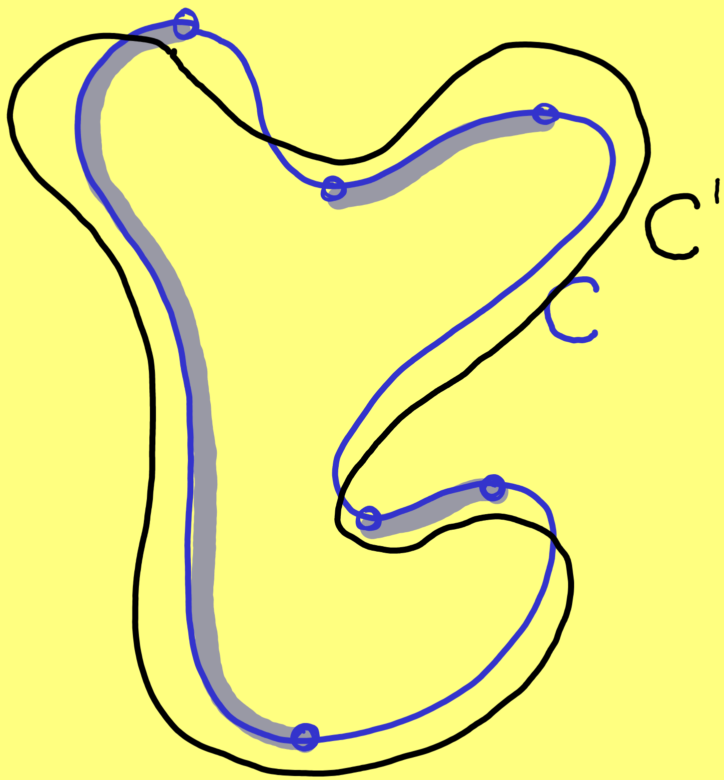

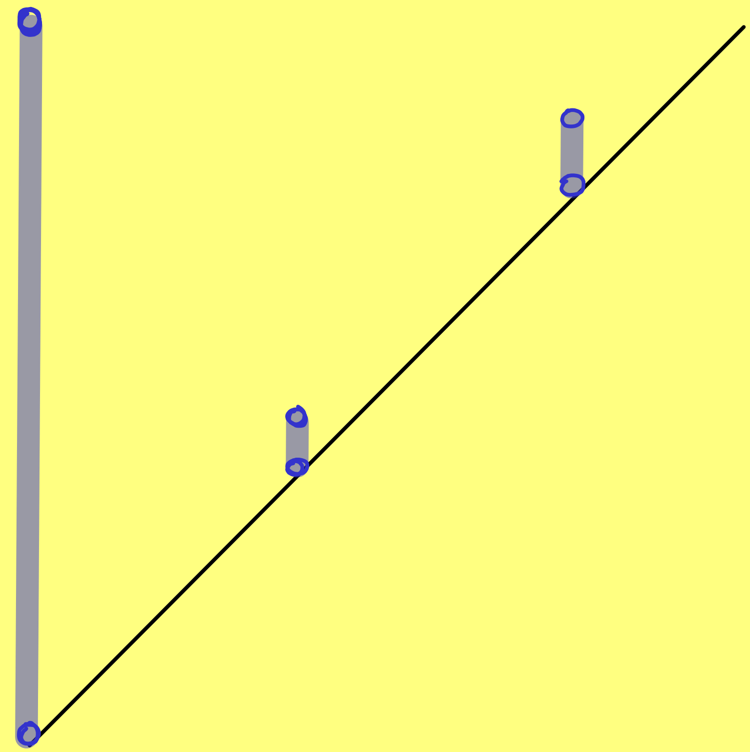

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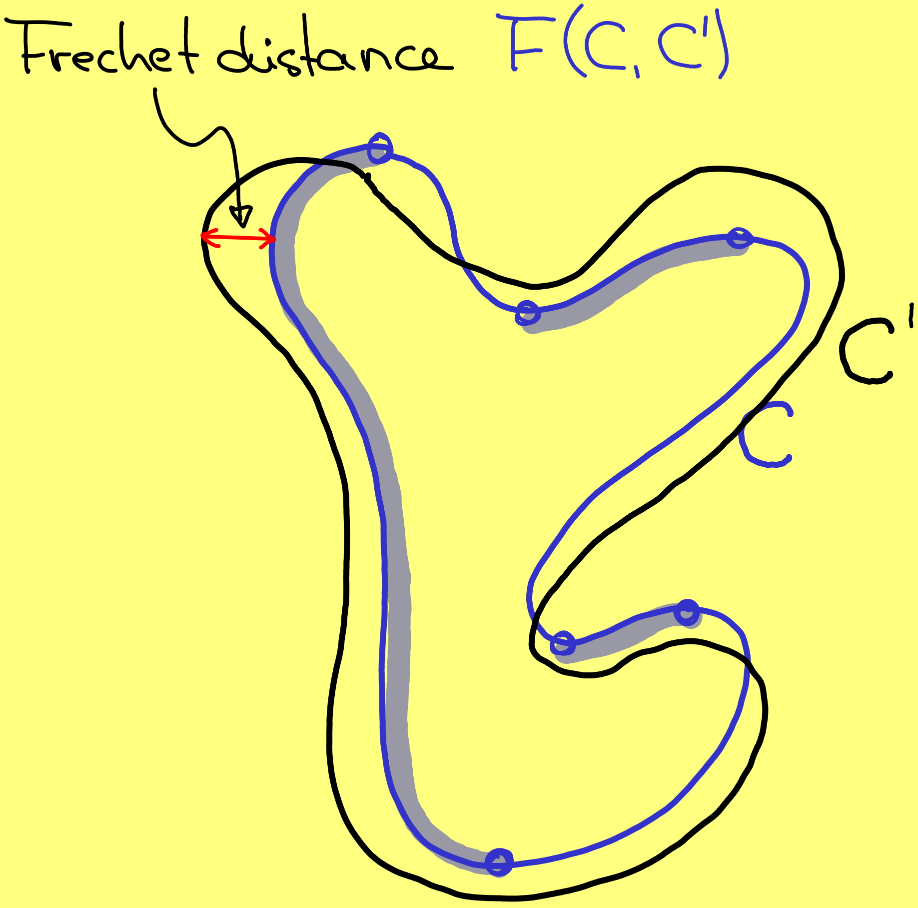

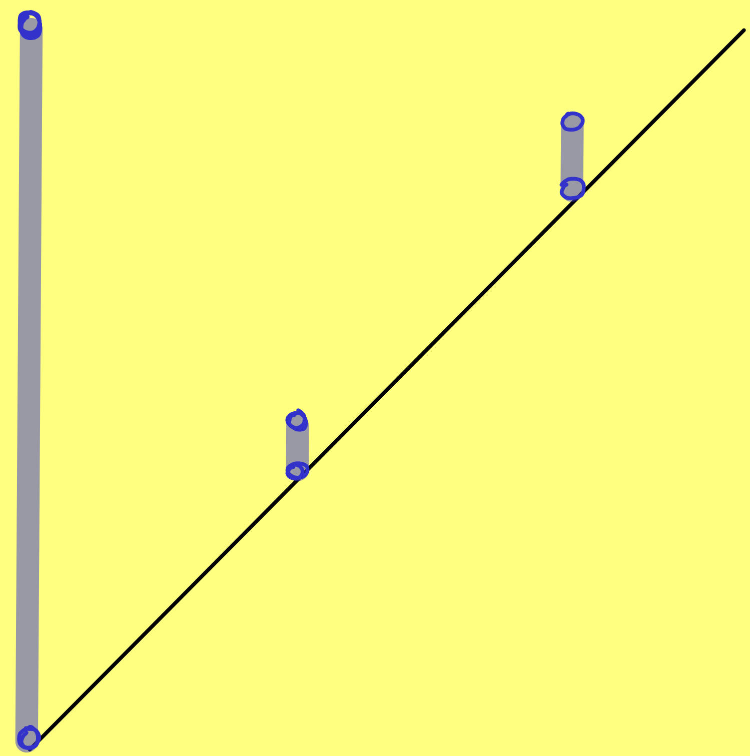

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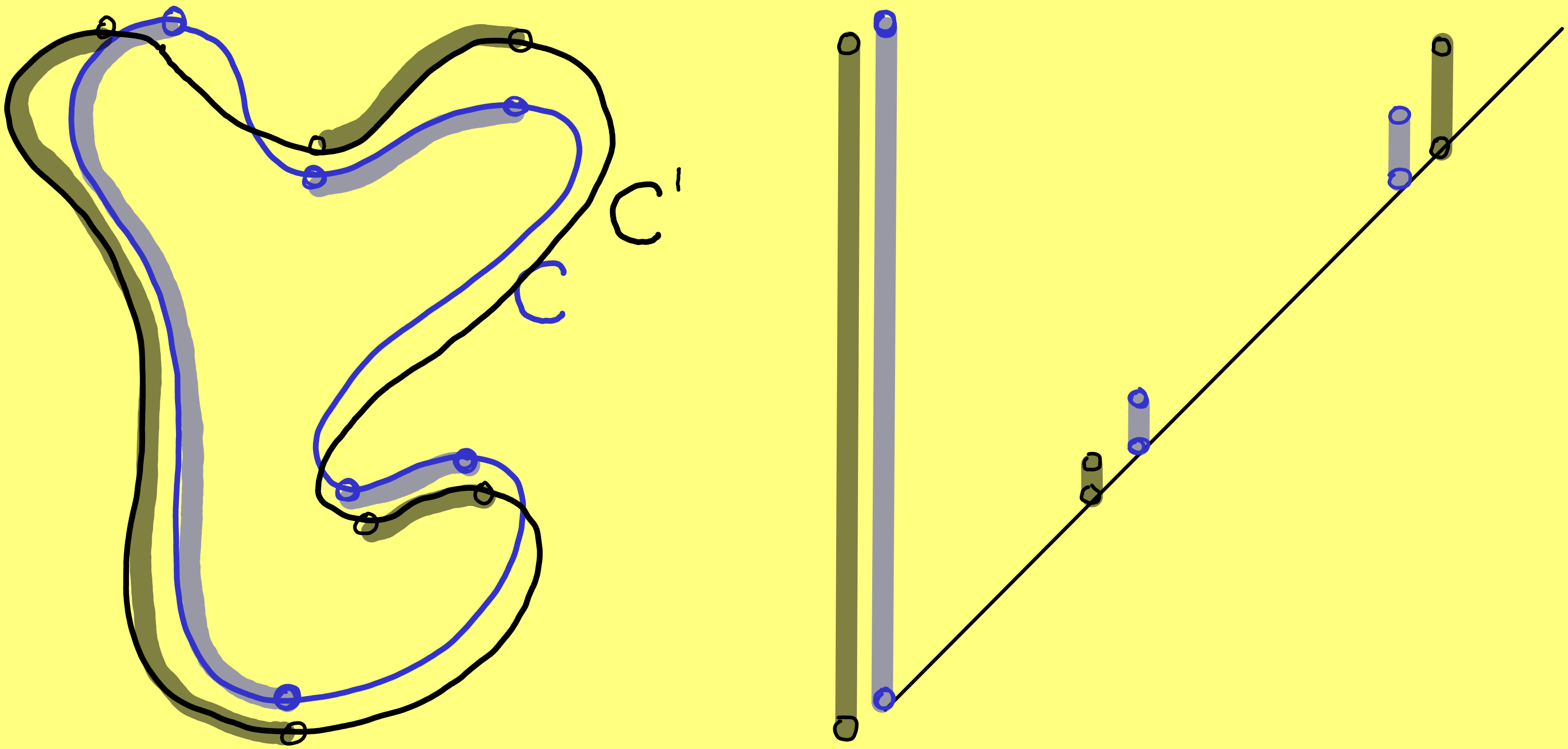

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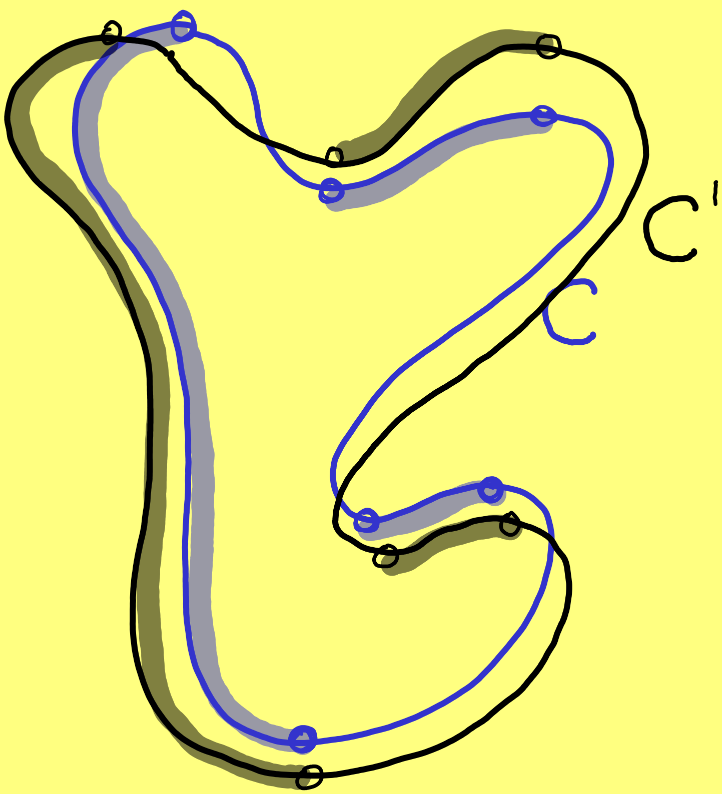

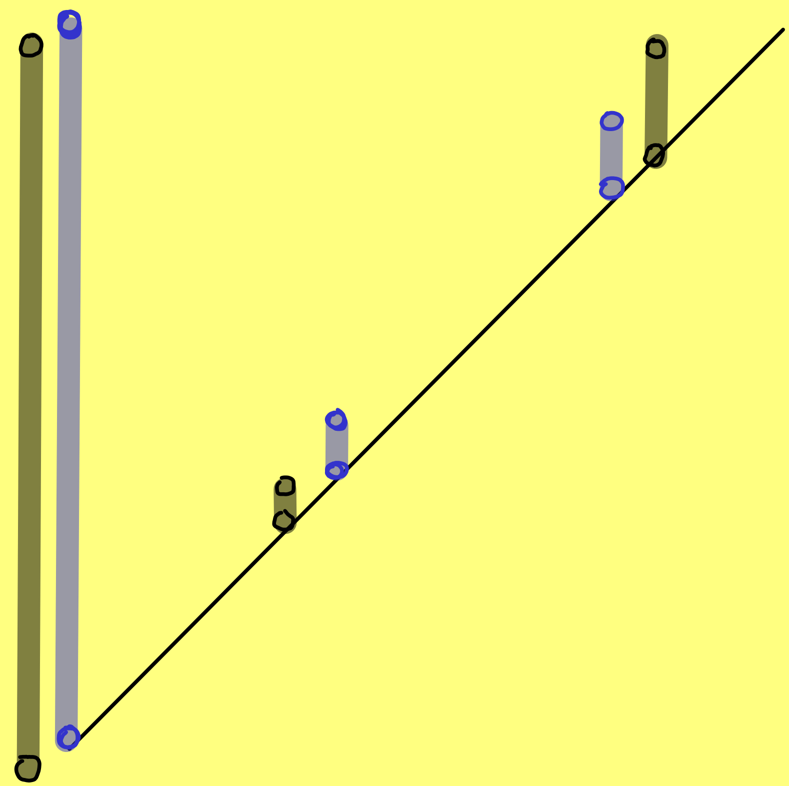

Thm. $|L(C) - L(C')| \leq (K(C) + K(C') - 2\pi)\, F(C, C')$.

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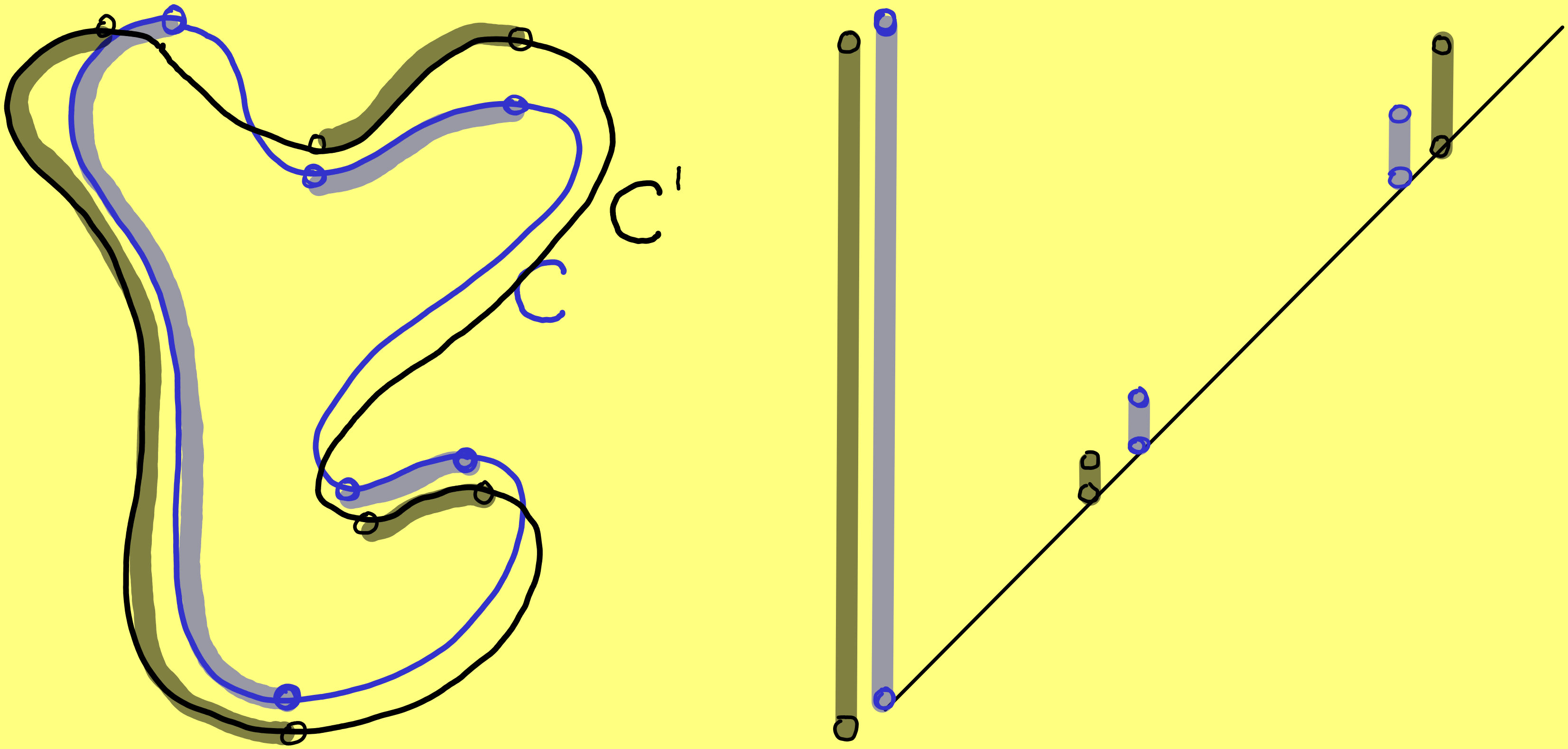

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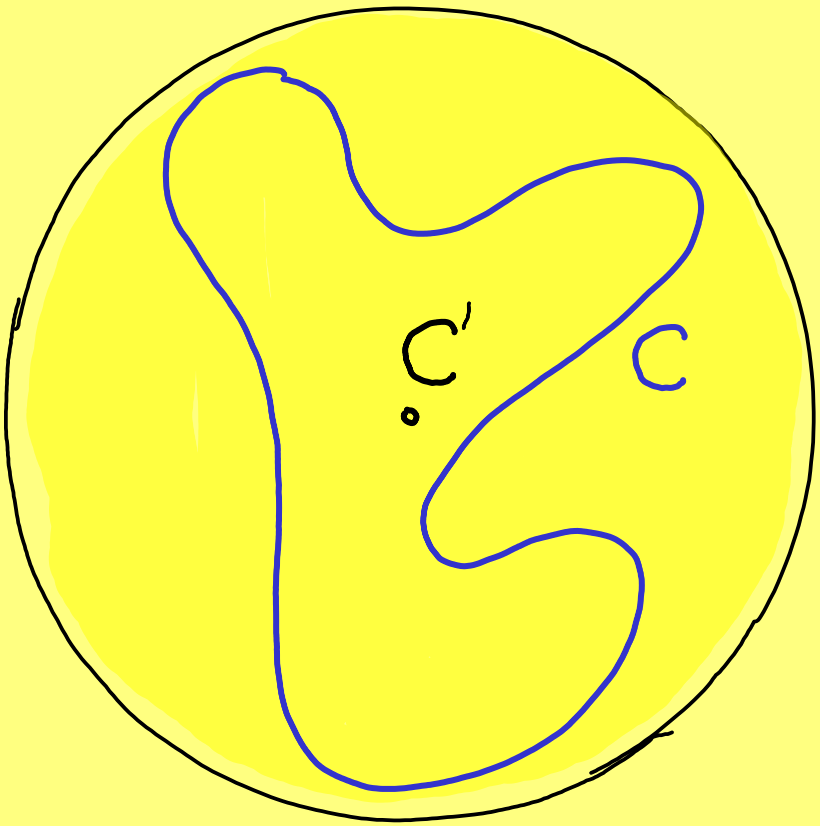

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I Persistence

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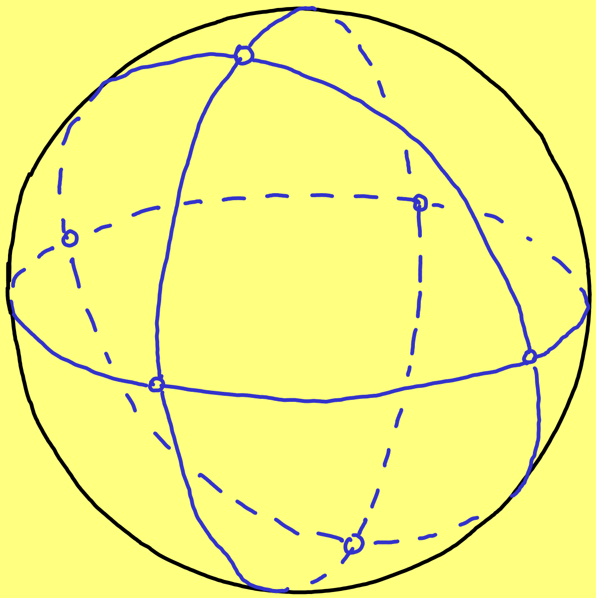
III Somites

III.1 Size and Growth



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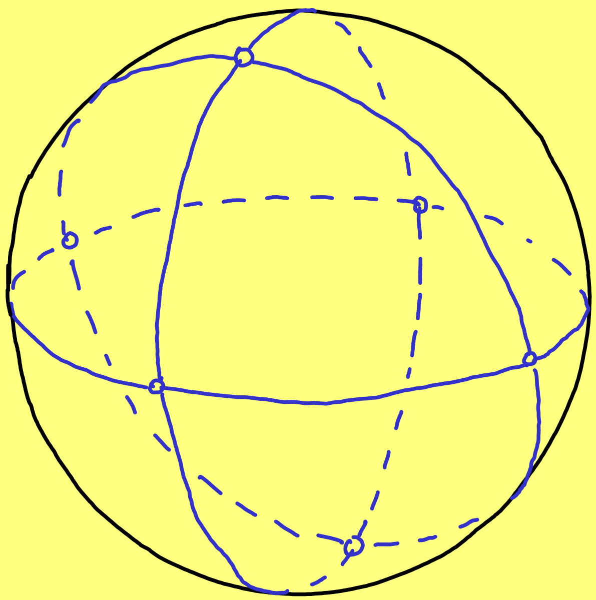

mesh (K) = max. diameter of a simplex

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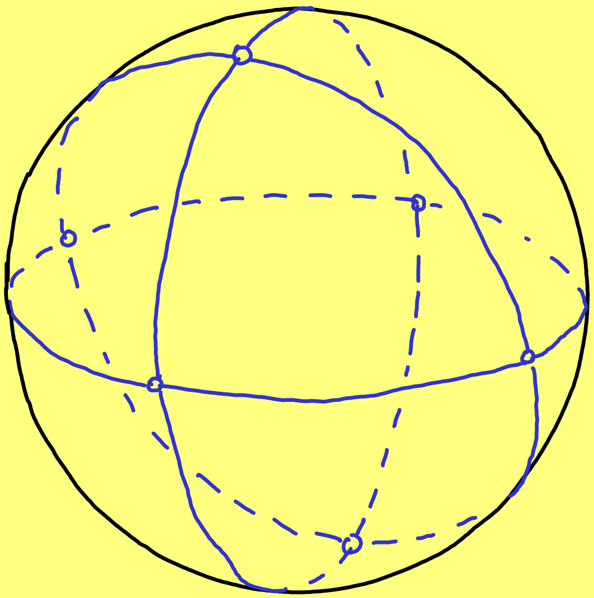

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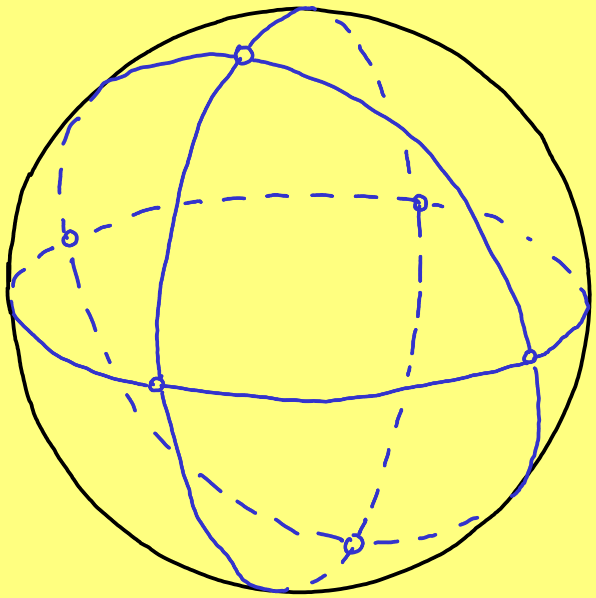

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The triangulation grows polynomially if $\exists$ constants C, M s.t.

$$N(r) \le C/r^M.$$

III.2 Total Persistence

The q-th total persistence of f is

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f Lipschitz and $N(r) \le C/r^M$

$\Longrightarrow$ $\mathrm{Pers}_q(f) \le \mathrm{const}$ for all $q > M$

III.2 Total Persistence

TP-Stab. Thm. Let $\mathbb{X}$ be a compact metric space with polynomially growing triangulation and $f, g : \mathbb{X} \to \mathbb{R}$ Lipschitz functions. Then

$$\mathrm{Pers}_q(f) - \mathrm{Pers}_q(g) \leq \mathrm{const} \cdot \|f-g\|_\infty$$

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2009 Cohen-Steiner, E., Harer, Mileyko

III.3 SIMPLIFICATION

normalized to $\mathrm{amp}(f) = \max_x f(x) - \min_x f(x) = 1$

III.3 Simplification

An ε-simplification of f is a function $f_\varepsilon : \mathbb{X} \to \mathbb{R}$ with $\|f - f_\varepsilon\|_\infty \le \varepsilon$ whose diagrams $\mathrm{Dgm}_p(f_\varepsilon)$ are same as $\mathrm{Dgm}_p(f)$ without points of persistence at most ε.

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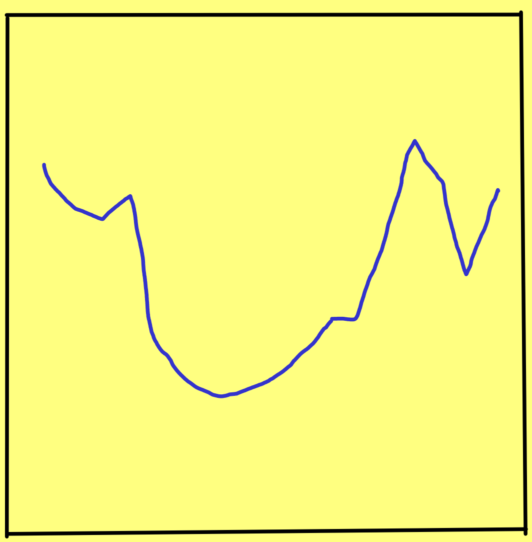

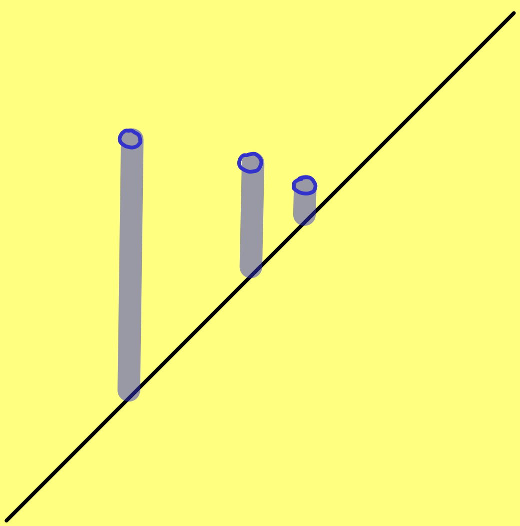

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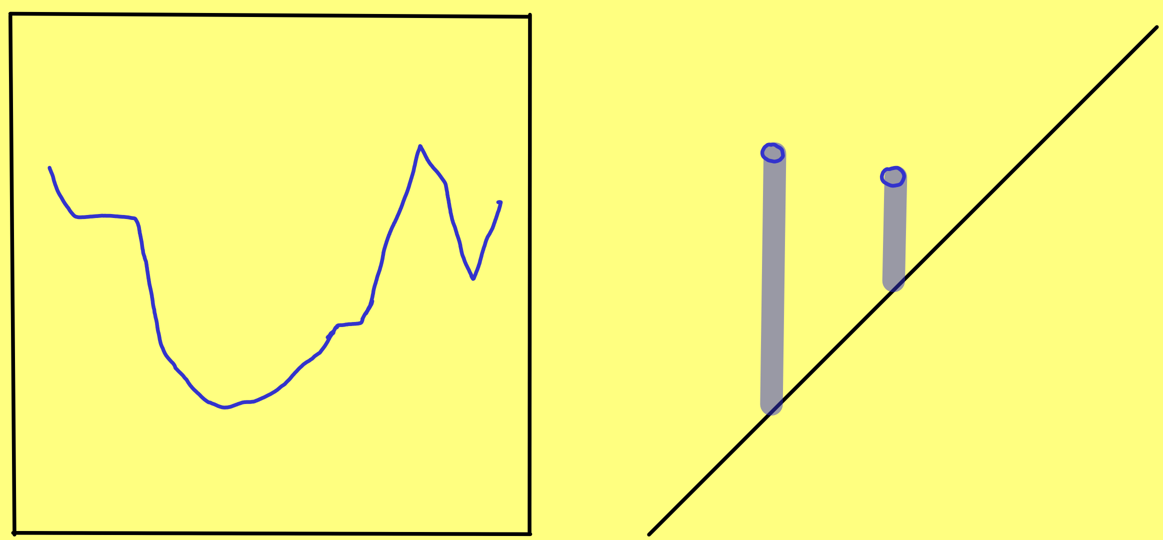

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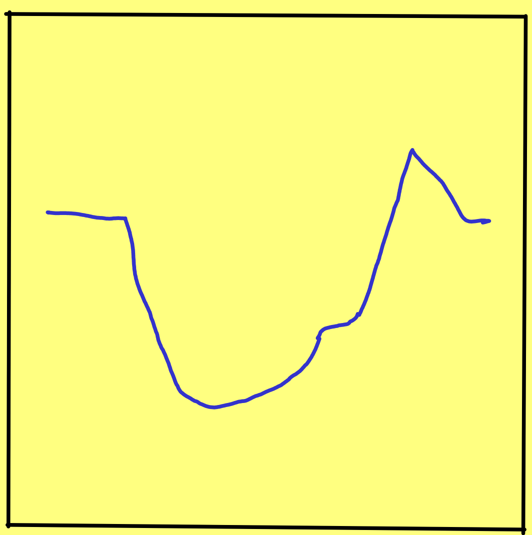

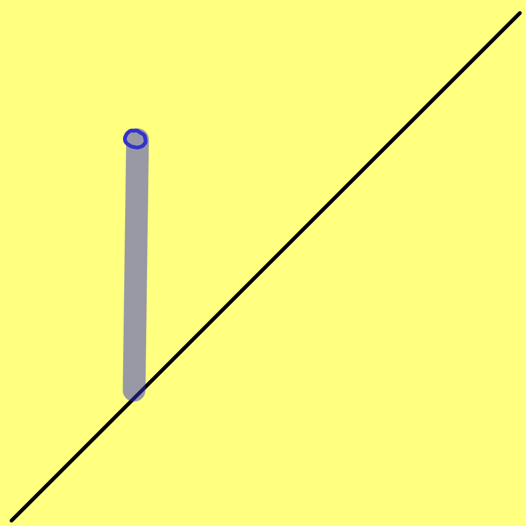

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III.3 SIMPLIFICATION

$f : \mathbb{S}^1 \to \mathbb{R}$

$$M_0(f) = \frac{1}{2} \#\text{crit. pts.} = \mathrm{Pers}_0(f)$$

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etc.

III.3 SIMPLIFICATION

$f : \mathbb{S}^1 \to \mathbb{R}$

not stable

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stable

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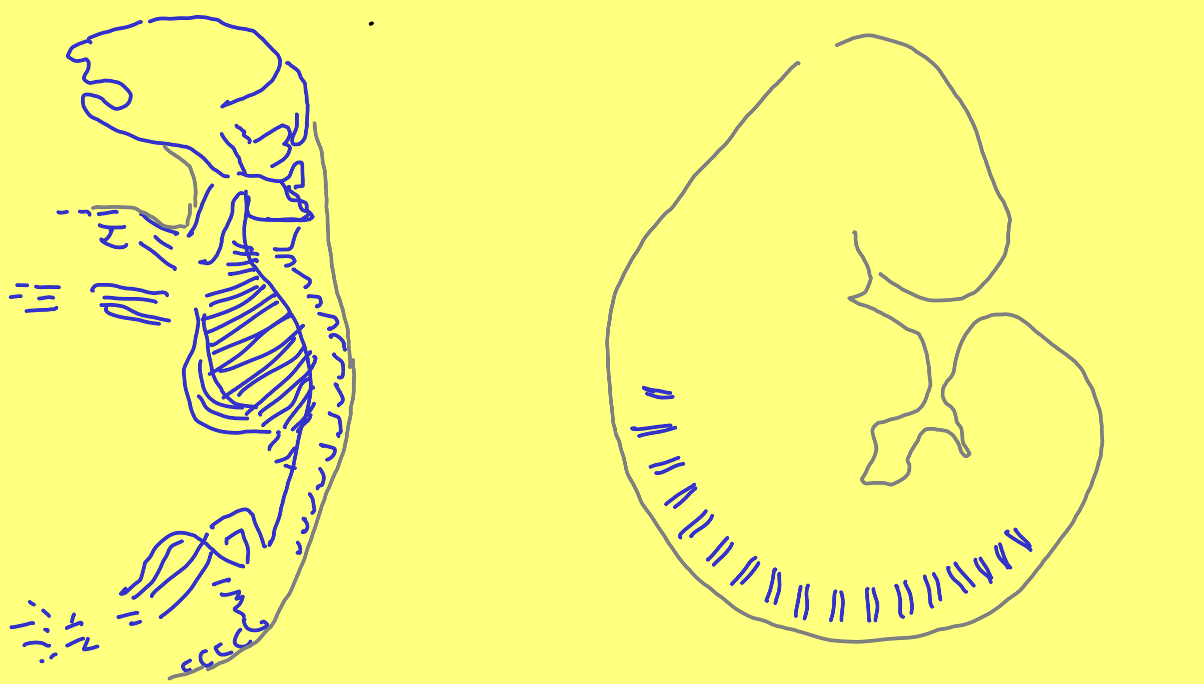
etc.

III.4 Rhythmic Gene Expression

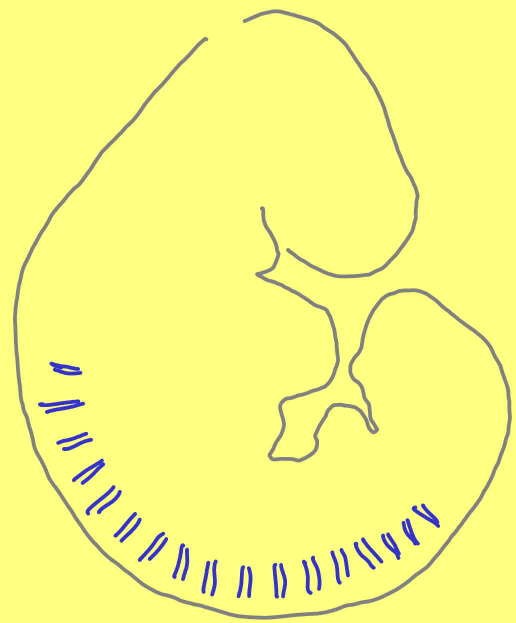


somite development is a rythmic process

III.4 RHYTHMIC GENE EXPRESSION

adult mouse

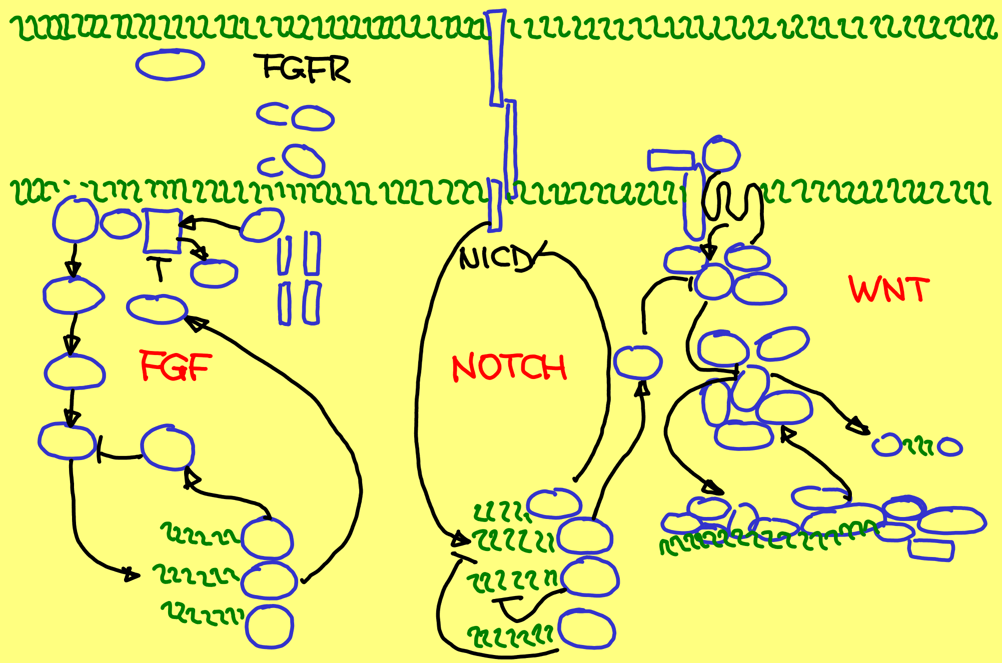
mouse embryo



somite development is a rythmic process

Pourqie Lab, Stowers

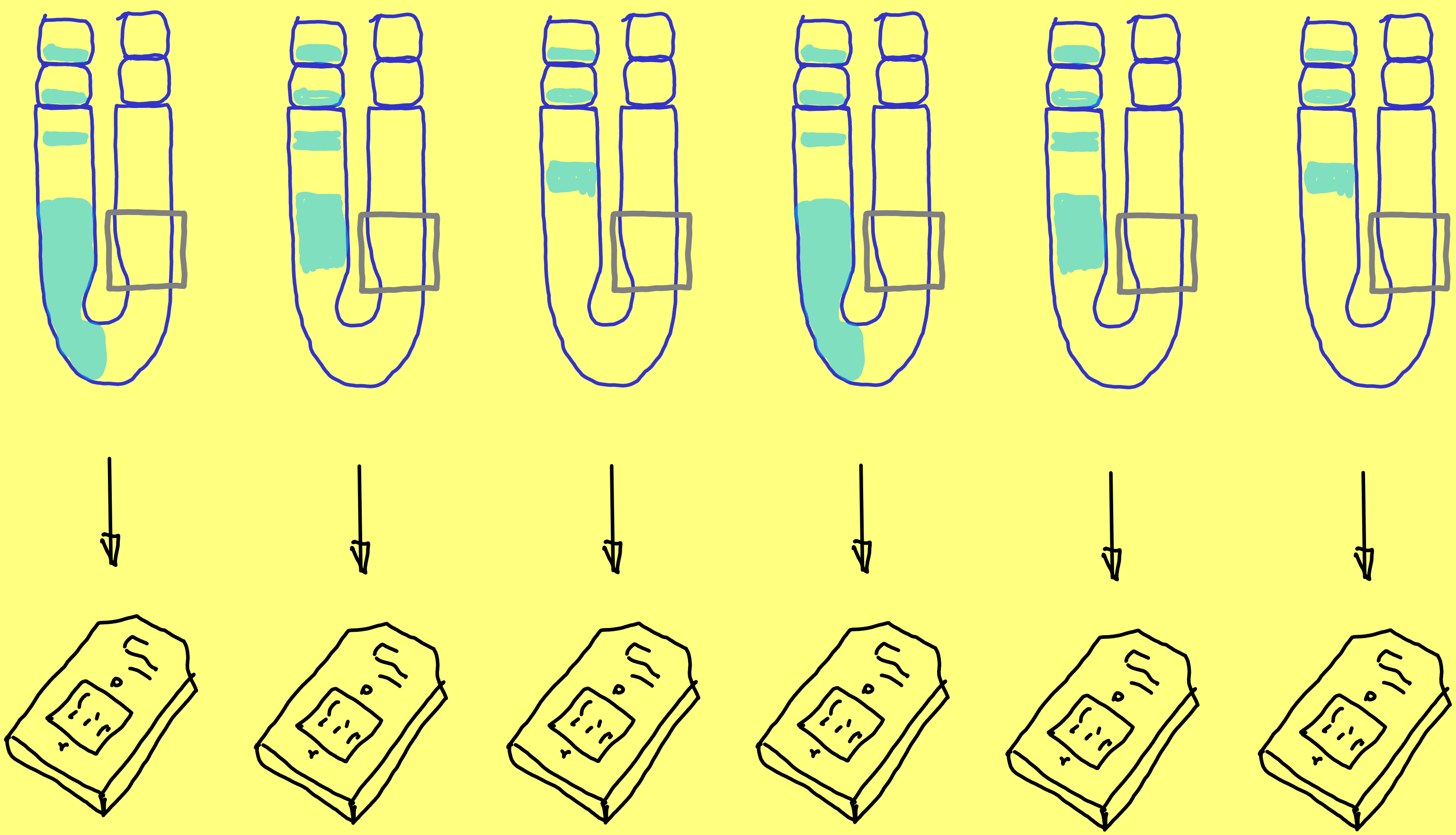
III.4 Rhythmic Gene Expression



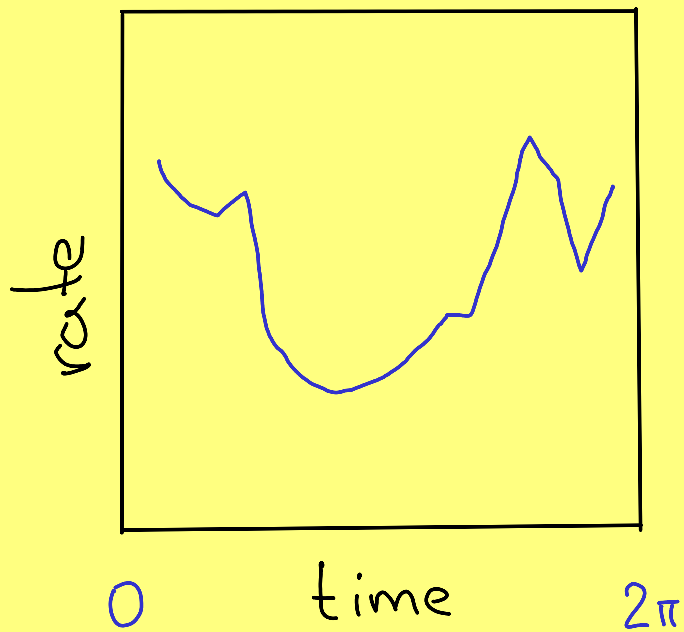
the proposed clock

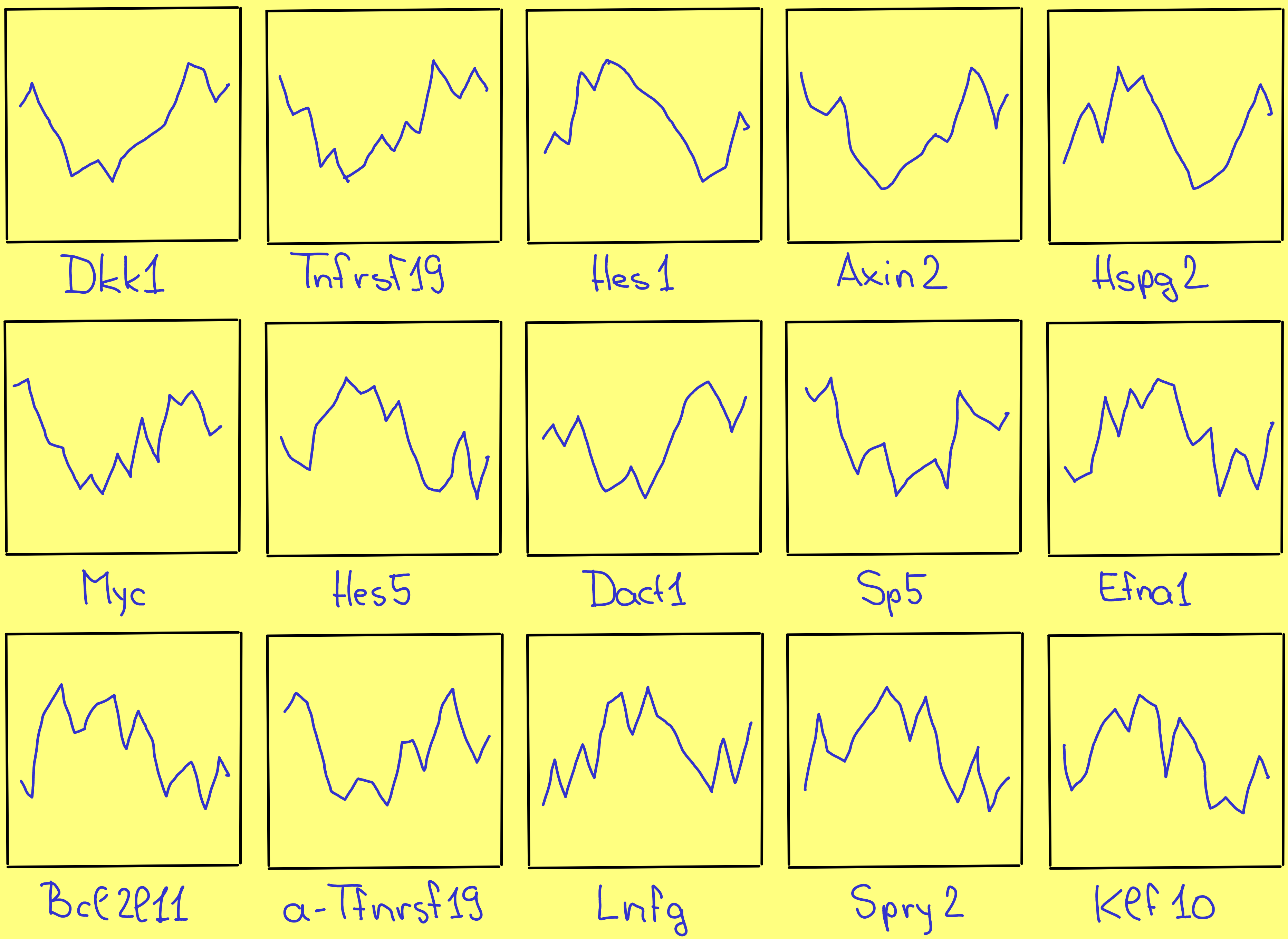
III.4 Rhythmic Gene Expression

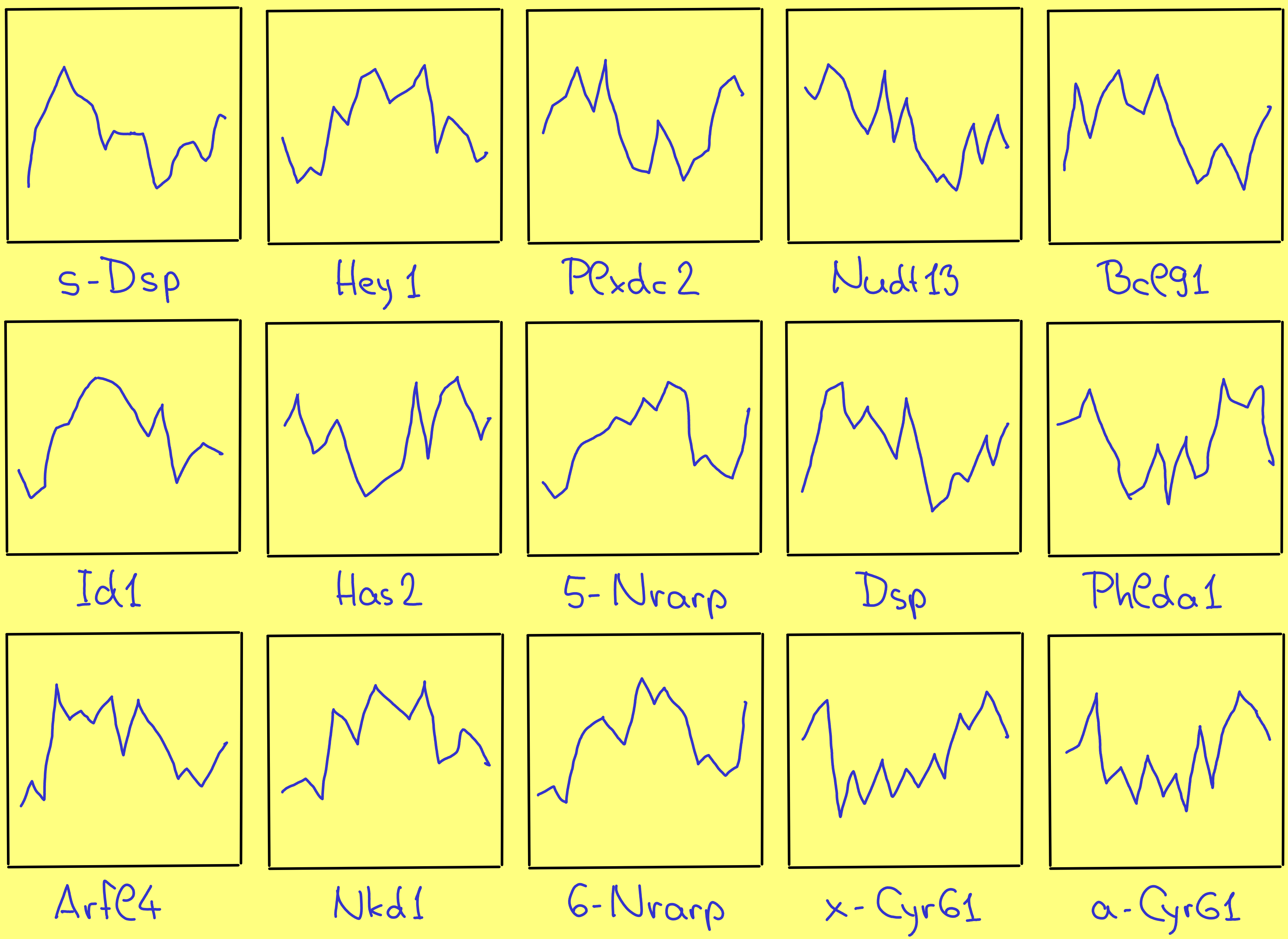
... time series microarray analysis



III.4 Rhythmic Gene Expression



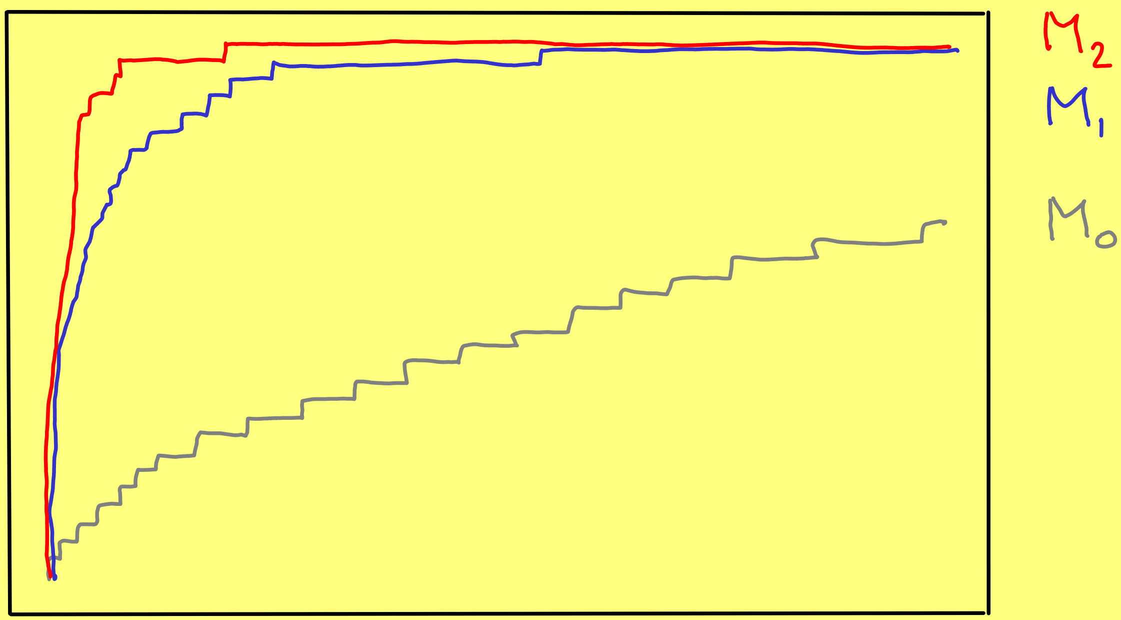

Dkk1
Tnfrsf19
Hes1
Axin2
Hspg2
Myc
Hes5
Dact1
Sp5
Efna1
Bcl2l11
a-Tfnrsf19
Lnfg
Spry2
Klf10


s-Dsp
Hey1
Plxdc2
Nudt13
Bcl9l
Id1
Has2
5-Nrarp
Dsp
Phlda1
Arl4
Nkd1
6-Nrarp
x-Cyr61
a-Cyr61

III.4 Rhythmic Gene Expression

yield of 30 biologically confirmed genes

ROC curves



	M_0	M_1	M_2	M_3	M_4	max
Area	4.42	9.75	10.18	10.19	10.15	10.50

THANK YOU